Partial Differential Equations

Part I

Lecture Notes

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Contents

1 Introduction **7**

1.1 Examples . . . 10

1.2 Equations from variational problems . . . 13

1.2.1 Ordinary differential equations . . . 13

1.2.2 Partial differential equations . . . 15

1.3 Exercises . . . 21

2 Equations of first order **25**

2.1 Linear equations . . . 25

2.2 Quasilinear equations . . . 31

2.2.1 A linearization method . . . 32

2.2.2 Initial value problem of Cauchy . . . 33

2.3 Nonlinear equations in two variables . . . 41

2.3.1 Initial value problem of Cauchy . . . 49

2.4 Nonlinear equations in $\mathbb{R}^n$ . . . 52

2.5 Hamilton-Jacobi theory . . . 54

2.6 Exercises . . . 61

3 Classification **65**

3.1 Linear equations of second order . . . 65

3.1.1 Normal form in two variables . . . 71

3.2 Quasilinear equations of second order . . . 75

3.2.1 Quasilinear elliptic equations . . . 75

3.3 Systems of first order . . . 76

3.3.1 Examples . . . 78

3.4 Systems of second order . . . 84

3.4.1 Examples . . . 85

3.5 Theorem of Cauchy and Kovalevski . . . 86

3.5.1 Appendix: Real analytic functions 93
3.6 Exercises 105

4 Hyperbolic equations 111
4.1 One dimensional wave equation 111
4.2 Higher dimensions 114
4.2.1 Case n=3 116
4.2.2 Case $n = 2$ 119
4.3 Inhomogeneous equation 122
4.4 Method of Riemann 125
4.5 Initial-boundary value problems 130
4.5.1 Oscillation of a string 130
4.5.2 Oscillation of a membran 133
4.5.3 Examples 134
4.5.4 Inhomogeneous wave equations 137
4.6 Exercises 142

5 Fourier transform 147
5.1 Definition, properties 147
5.1.1 Pseudodifferential operators 152
5.2 Exercices 155

Preface

These lecture notes are indented as a straightforward introduction to partial differential equations which can serve as a textbook for undergraduate and beginning graduate students.

For additional reading we recommend following books: W. I. Smirnov [17], I. G. Petrowski[13], W. A. Strauss [19], F. John [8], L. C. Evans [5] and R. Courant and D. Hilbert [4]. Some material of these lecture notes was taken from some of these books.

Chapter 1

Introduction

Ordinary and partial differential equations occur in many applications. An ordinary differential equation is a special case of a partial differential equation but the behaviour of solutions is quite different in general. It is much more complicated in the case of partial differential equations caused by the fact that the functions for which we are looking at are functions of more than one independent variable.

Equation

$$F(x, y(x), y'(x), \ldots, y^{(n)}) = 0$$

is an *ordinary differential equation* of n-th order for the unknown function $y(x)$, where F is given.

An important problem for ordinary differential equations is the *initial value problem*

$$\begin{aligned} y'(x) &= f(x, y(x)) \\ y(x_0) &= y_0 \ , \end{aligned}$$

where f is a given real function of two variables $x,\ y$ and $x_0,\ y_0$ are given real numbers.

Picard-Lindelöf Theorem. *Suppose*
(i) $f(x,y)$ *is continuous in a rectangle*

$$Q = \{(x,y) \in \mathbb{R}^2 :\ |x - x_0| < a,\ |y - y_0| < b\}.$$

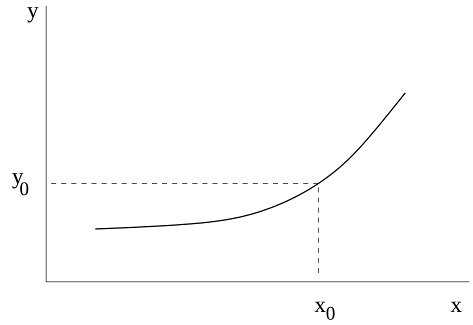

Figure 1.1: Initial value problem

(ii) There is a constant K such that $|f(x,y)| \leq K$ for all $(x,y) \in Q$.
(ii) Lipschitz condition: There is a constant L such that

$$|f(x,y_2) - f(x,y_1)| \leq L|y_2 - y_1|$$

for all $(x,y_1),(x,y_2)$.

Then, there exists a unique solution $y \in C^1(x_0-\alpha, x_0+\alpha)$ of the above initial value problem, where $\alpha = \min(b/K, a)$.

The linear ordinary differential equation

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots a_1(x)y' + a_0(x)y = 0,$$

where a_j are continuous functions, has exactly n linearly independent solutions. In contrast to this property the partial differential $u_{xx} + u_{yy} = 0$ in $\mathbb{R}^2$ has infinitely many linearly independent solutions in the linear space $C^2(\mathbb{R}^2)$.

For the ordinary differential equation of second order

$$y''(x) = f(x, y(x), y'(x))$$

there exist in general a family of solutions with two free parameters. Thus, it is naturally to consider the associated *initial value problem*

$$\begin{aligned} y''(x) &= f(x, y(x), y'(x)) \\ y(x_0) &= y_0,\ y'(x_0) = y_1, \end{aligned}$$

where y_0 and y_1 are given, or to consider the *boundary value problem*

$$\begin{aligned} y''(x) &= f(x, y(x), y'(x)) \\ y(x_0) &= y_0,\ y(x_1) = y_1. \end{aligned}$$

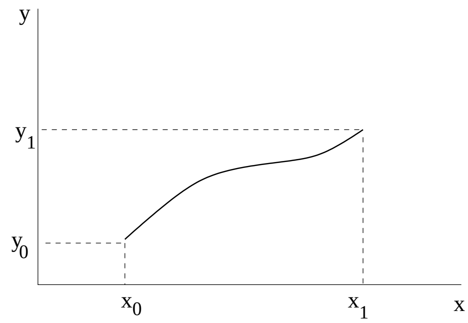

Figure 1.2: Boundary value problem

Initial and boundary value problems play also an important role in the theory of partial differential equations. A *partial differential equation* for the unknown function $u(x, y)$ is for example

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0,$$

where the function F is given. This equation is of second order.

An equation is said to be of *n-th order* if the highest derivative which occurs are of order n.

An equation is said to be *linear* if the unknown function and its derivatives are linear in F. For example,

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y),$$

where the functions $a,\ b,\ c$ and f are given, is a linear equation of first order.

An equation is said to be *quasilinear* if the highest derivatives occur linearly in the equation. For example,

$$a(x, y, u, u_x, u_y)u_{xx} + b(x, y, u, u_x, u_y)u_{xy} + c(x, y, u, u_x, u_y)u_{yy} = 0$$

is a quasilinear equation of second order.

1.1 Examples

1. $u_y = 0$, where $u = u(x,y)$. All functions $u = w(x)$ are solutions.

2. $u_x = u_y$, where $u = u(x,y)$. A change of coordinates transforms this equation into an equation of the first example. Set $\xi = x + y,\ \eta = x - y$, then

$$u(x,y) = u\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) =: v(\xi,\eta).$$

Assume $u \in C^1$, then

$$v_\eta = \frac{1}{2}(u_x - u_y).$$

If $u_x = u_y$, then $v_\eta = 0$ and vice versa, thus $v = w(\xi)$ are solutions for arbitrary C^1-functions $w(\xi)$. Consequently, we have a large class of solutions of the original partial differential equation: $u = w(x + y)$ with an *arbitrary* C^1-function w.

3. A necessary and sufficient condition that for given C^1-functions $M,\ N$ the integral

$$\int_{P_0}^{P_1} M(x,y)dx + N(x,y)dy$$

is independent of the curve which connects the points P_0 with P_1 in a simply connected domain $\Omega \subset \mathbb{R}^2$ is the partial differential equation (condition of integrability)

$$M_y = N_x$$

in Ω.

This is one equation for two functions. A large class of solutions are given by $M = \Phi_x,\ N = \Phi_y$, where $\Phi(x,y)$ is an arbitrary C^2-function. It follows from Gauss theorem that these are all C^1-solutions of the above differential equation.

4. *Method of an integrating multiplier for an ordinary differential equation.* Consider the ordinary differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

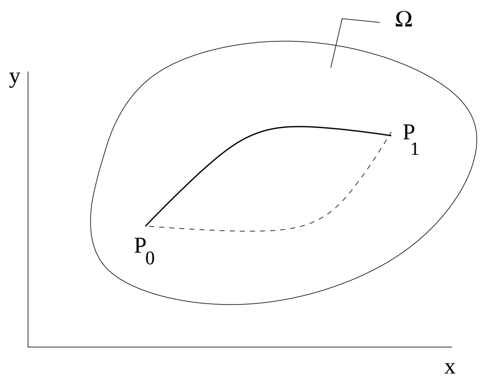

Figure 1.3: Independence of the path

for given C^1-functions $M,\ N$. Then we seek a C^1-function $\mu(x,y)$ such that $\mu M dx + \mu N dy$ is a total differential, that is, that $(\mu M)_y = (\mu N)_x$ is satisfied. This is a linear partial differential equation of first order for μ:

$$M\mu_y - N\mu_x = \mu(N_x - M_y).$$

5. Two C^1-functions $u(x,y)$ and $v(x,y)$ are said to be *functionally dependent* if

$$\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = 0,$$

that is, if

$$u_x v_y - u_y v_x = 0.$$

This is a linear partial differential equation of first order for u if v is a given C^1-function. A large class of solutions is given by

$$u = H(v(x,y)),$$

where H is an *arbitrary* C^1-function.

6. *Cauchy-Riemann equations.* Set $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$ and $u,\ v$ are given $C^1(\Omega)$-functions. Here is Ω a domain in $\mathbb{R}^2$. If the function $f(z)$ is differentiable with respect to the complex variable z then $u,\ v$ satisfy the Cauchy-Riemann equations

$$u_x = v_y,\quad u_y = -v_x.$$

It is known from the theory of functions of one complex variable that the real part u and the imaginary part v of a differentiable function $f(z)$ are solutions of the *Laplace equation*

$$\triangle u = 0, \quad \triangle v = 0,$$

where $\triangle u = u_{xx} + u_{yy}$.

7. The *Newton potential*

$$u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

is a solution of the Laplace equation in $\mathbb{R}^3 \setminus (0,0,0)$, that is, of

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

8. *Heat equation.* Let $u(x,t)$ be the temperature of a point $x \in \Omega$ at time t, where $\Omega \subset \mathbb{R}^3$ is a domain. Then $u(x,t)$ satisfies in $\Omega \times [0, \infty)$ the *heat equation*

$$u_t = k \triangle u,$$

where $\triangle u = u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}$ and k is a positive constant. The condition

$$u(x,0) = u_0(x), \quad x \in \Omega,$$

where $u_0(x)$ is given, is an *initial condition* associated to the above heat equation. The condition

$$u(x,t) = h(x,t), \quad x \in \partial\Omega, \ t \geq 0,$$

where $h(x,t)$ is given is a *boundary condition* for the heat equation.

If $h(x,t) = g(x)$, that is, h is independent of t, then one expects that the solution $u(x,t)$ tends to a function $v(x)$ independent of t if $t \to \infty$. Moreover, it turns out that v is the solution of the *boundary value problem* for the Laplace equation

$$\begin{aligned} \triangle u &= 0 \text{ in } \Omega \\ u &= g(x) \text{ on } \partial\Omega. \end{aligned}$$

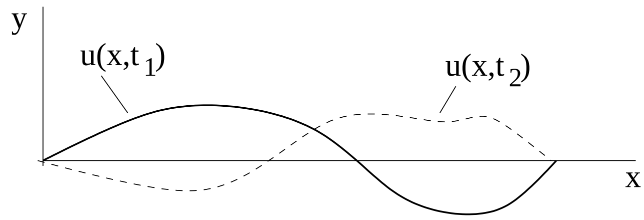

Figure 1.4: Oscillating string

9. *Wave equation.* The wave equation

$$u_{tt} = c^2 \triangle u,$$

where $u = u(x,t)$ and c is a positive constant, describes, for example, oscillations of membranes or of three dimensional domains. In the one dimensioal case

$$u_{tt} = c^2 u_{xx}$$

describes oscillations of a string, for example.

Associated *initial conditions* are

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x),$$

where u_0, u_1 are given functions. That is, the initial position and the initial velocity are prescribed.

If the string is finite one describes additionally *boundary conditions*, for example

$$u(0,t) = 0, \quad u(l,t) = 0 \ \text{ for all } t \ge 0.$$

1.2 Equations from variational problems

A large class of ordinary and partial differential equations arise from variational problems.

1.2.1 Ordinary differential equations

Set

$$E(v) = \int_a^b f(x, v(x), v'(x))\ dx$$

and for given $u_a,\ u_b \in \mathbb{R}$

$$V = \{v \in C^2[a,b] : \ v(a) = u_a,\ v(b) = u_b\},$$

where $-\infty < a < b < \infty$ and f is sufficiently regular. One of the basic problems in the calculus of variation is

$$(P) \qquad \min_{v \in V} E(v).$$

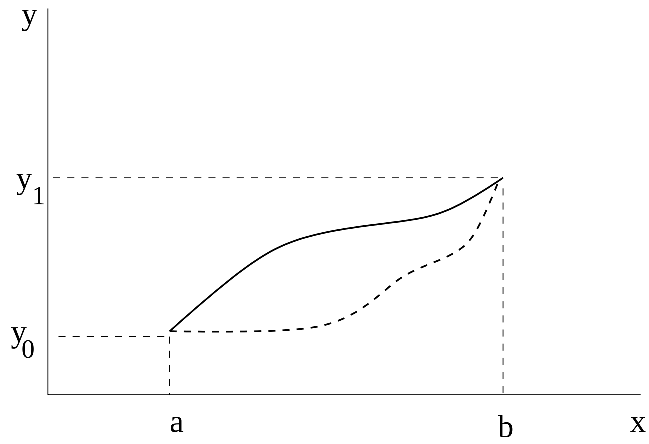

Figure 1.5: Admissible variations

Euler equation. *Let $u \in V$ be a solution of (P), then*

$$\frac{d}{dx} f_{u'}(x, u(x), u'(x)) = f_u(x, u(x), u'(x))$$

in (a,b).

Proof. Exercise. Hints: For fixed $\phi \in C^2[a,b]$ with $\phi(a) = \phi(b) = 0$ and real ϵ, $|\epsilon| < \epsilon_0$, set $g(\epsilon) = E(u + \epsilon\phi)$. Since $g(0) \le g(\epsilon)$ it follows $g'(0) = 0$. Integration by parts in the formula for $g'(0)$ and the following basic lemma in the calculus of variations imply Euler equation.

Basic lemma in the calculus of variations. *Let $h \in C(a,b)$ and*

$$\int_a^b h(x)\phi(x)\ dx = 0$$

for all $\phi \in C_0^1(a,b)$. Then $h(x) = 0$ on (a,b).

Proof. Assume $h(x_0) > 0$ for an $x_0 \in (a,b)$, then there is a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (a,b)$ and $h(x) \geq h(x_0)/2$ on $(x_0 - \delta, x_0 + \delta)$. Set

$$\phi(x) = \begin{cases} (\delta^2 - |x - x_0|^2)^2 & \text{if} \quad x \in (x_0 - \delta, x_0 + \delta) \\ 0 & \text{if} \quad x \in (a,b) \setminus [x_0 - \delta, x_0 + \delta] \end{cases}.$$

Thus $\phi \in C_0^1(a,b)$ and

$$\int_a^b h(x)\phi(x)\ dx \geq \frac{h(x_0)}{2} \int_{x_0-\delta}^{x_0+\delta} \phi(x)\ dx > 0,$$

which is a contradiction to the assumption of the lemma.

1.2.2 Partial differential equations

The same procedure as above applied to the following multiple integral leads to a second order quasilinear partial differential equation. Set

$$E(v) = \int_\Omega F(x, v, \nabla v)\ dx,$$

where $\Omega \subset \mathbb{R}^n$ is a domain, $x = (x_1, \ldots, x_n)$, $v = v(x) : \ \Omega \mapsto \mathbb{R}$, and $\nabla v = (v_{x_1}, \ldots, v_{x_n})$. It is assumed that the function F is sufficiently regular in its arguments. For a given function h, defined on $\partial\Omega$, set

$$V = \{v \in C^2(\overline{\Omega}) : \ v = h \text{ on } \partial\Omega\}.$$

Euler equation. *Let $u \in V$ be a solution of (P), then*

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} F_{u_{x_i}} - F_u = 0$$

in Ω.

Proof. Exercise. Hint: Extend the above fundamental lemma of the calculus of variations to the case of multiple integrals. The interval $(x_0 - \delta, x_0 + \delta)$ in the definition of ϕ must be replaced by a ball with center at x_0 and radius δ.

Example: **Dirichlet integral**

In two dimensions the Dirichlet integral is given by

$$D(v) = \int_\Omega \left(v_x^2 + v_y^2\right)\ dxdy$$

and the associated Euler equation is the Laplace equation $\triangle u = 0$ in Ω.

Thus, there is natural relationship between the boundary value problem

$$\triangle u = 0 \ \text{ in } \Omega,\ \ u = h \ \text{ on } \ \partial\Omega$$

and the variational problem

$$\min_{v \in V}\ D(v).$$

But these problems are not equivalent in general. It can happen that the boundary value problem has a solution but the variational problem has no solution, see for an example Courant and Hilbert [4], Vol. 1, p. 155, where h is a continuous function and the associated solution u of the boundary value problem has no finite Dichlet integral.

The problems are equivalent, provided the given boundary value function h is in the class $H^{1/2}(\partial\Omega)$, see Lions and Magenes [11].

Example: **Minimal surface equation**

The non-parametric minimal surface problem in two dimensions is to find a minimizer $u = u(x_1, x_2)$ of the problem

$$\min_{v \in V} \int_\Omega \sqrt{1 + v_{x_1}^2 + v_{x_2}^2}\ dx,$$

where for given function h defined on the boundary of the domain Ω

$$V = \{v \in C^1(\overline{\Omega}) :\ v = h \text{ on } \partial\Omega\}.$$

Suppose that the minimizer satisfies the regularity assumption $u \in C^2(\Omega)$, then u is a solution of the *minimal surface equation* (Euler equation) in Ω

$$\frac{\partial}{\partial x_1}\left(\frac{u_{x_1}}{\sqrt{1 + |\nabla u|^2}}\right) + \frac{\partial}{\partial x_2}\left(\frac{u_{x_2}}{\sqrt{1 + |\nabla u|^2}}\right) = 0. \tag{1.1}$$

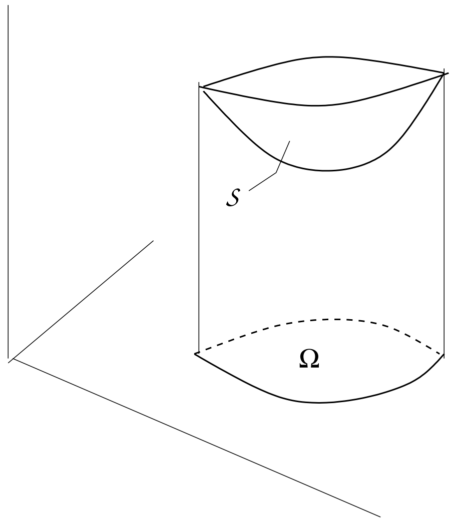

Figure 1.6: Minimal surface

In fact, the additional assumption $u \in C^2(\Omega)$ is superflous since it follows from regularity considerations for quasilinear elliptic equations of second order, see for example Gilbarg and Trudinger [7].

Let $\Omega = \mathbb{R}^2$. Each linear function is a solution of the minimal surface equation (1.1). It was shown by Bernstein [2] that these functions are *all* solutions of the minimal surface quation. This is true for higher dimensions $n \leq 7$, see Simons [15]. If $n \geq 8$, then there exists also other solutions which define cones, see Bombieri, Giust and De Giorgi [3].

The linearized minimal surface equation over $u \equiv 0$ is the Laplace equation $\triangle u = 0$. In $\mathbb{R}^2$ linear functions are solutions but also many other functions in contrast to the minimal surface equation. This striking difference is caused by the strong nonlinearity of the minimal surface equation.

More general minimal surfaces are described by using parametric representations. An example is shown in Figure 1.7[1]. See [14],pp. 62, for example, for rotationally symmetric minimal surfaces.

[1] An experiment from Beutelspacher's Mathematikum, Wissenschaftsjahr 2008, Leipzig

Figure 1.7: Rotationally symmetric minimal surface

Neumann type boundary value poblems

Set $V = C^1(\overline{\Omega})$ and

$$E(v) = \int_\Omega F(x, v, \nabla v)\ dx - \int_{\partial\Omega} g(x, v)\ ds,$$

where F and g are given sufficiently regular functions and $\Omega \subset \mathbb{R}^n$ is a bounded and sufficiently regular domain. Assume u is a minimizer of $E(v)$ in V, that is

$$u \in V: \quad E(u) \le E(v) \ \text{ for all } v \in V,$$

then

$$\begin{aligned} \int_\Omega \big(\sum_{i=1}^n F_{u_{x_i}}(x, u, \nabla u)\phi_{x_i} &+ F_u(x, u, \nabla u)\phi\big)\ dx \\ &- \int_{\partial\Omega} g_u(x, u)\phi\ ds = 0 \end{aligned}$$

for all $\phi \in C^1(\overline{\Omega})$. Assume additionally $u \in C^2(\Omega)$, then u is a solution of the Neumann type boundary value problem

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_{u_{x_i}} - F_u &= 0 \ \text{ in } \Omega \\ \sum_{i=1}^{n} F_{u_{x_i}} \nu_i - g_u &= 0 \ \text{ on } \partial\Omega, \end{aligned}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the exterior unit nornal at the boundary $\partial\Omega$. This follows after integration by parts from the basic lemma of the calculus of variations.

Example: **Laplace equation**

Set

$$E(v) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \int_{\partial\Omega} h(x) v \, ds,$$

then the associated boundary value problem is

$$\begin{aligned} \triangle u &= 0 \ \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} &= h \ \text{ on } \partial\Omega. \end{aligned}$$

Example: **Capillary equation**

Let $\Omega \subset \mathbb{R}^2$ and set

$$E(v) = \int_{\Omega} \sqrt{1 + |\nabla u|^2} \, dx + \frac{\kappa}{2} \int_{\Omega} v^2 \, dx - \cos\gamma \int_{\partial\Omega} v \, ds.$$

Here is κ a positive constant (capillarity constant) and γ is the (constant) boundary contact angle, that is, the angle between the container wall and the capillary surface defined by $u = u(x_1, x_2)$ at the boundary. Then, the related boundary value problem is

$$\begin{aligned} \text{d}iv\ (Tu) &= \kappa u \ \text{ in } \Omega \\ \nu \cdot Tu &= \cos\gamma \text{ on } \partial\Omega, \end{aligned}$$

where we use the abbreviation

$$Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}},$$

div (Tu) is equal to the left hand side of the minimal surface equation (1.1).

The above problem discribes the ascent of a liquid, water for example, in a vertical cylinder with cross section Ω. It is asumed that gravity is directed downward in the direction of the negative x_3-axis. Figure (1.8) shows that liquid can rise along a vertical wedge which is consequence of the strong nonlinearity of the underlying equations, see Finn [6]. This photo was taken

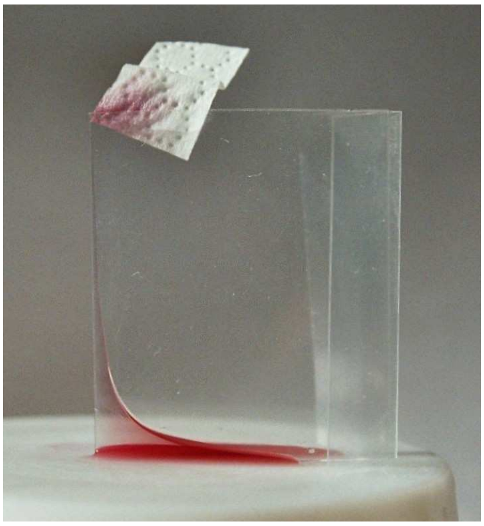

Figure 1.8: Ascent of liquid in a wedge

from [12].

1.3 Exercises

1. Find nontrivial solutions u of
$$u_x y - u_y x = 0 \ .$$

2. Prove: In the linear space $C^2(\mathbb{R}^2)$ there are infinitely many linearly independent solutions of $\triangle u = 0$ in $\mathbb{R}^2$.

   Hint. Real and imaginary part of holomorph functions are solutions of the Laplace equation.

3. Find all radially symmetric functions which satisfy the Laplace equation in $\mathbb{R}^n \setminus \{0\}$ for $n \geq 2$. A function u is said to be radially symmetric if $u(x) = f(r)$, where $r = (\sum_i^n x_i^2)^{1/2}$.

   Hint. Show that a radially symmetric u satisfies $\triangle u = r^{1-n}\left(r^{n-1}f'\right)'$ by using $\nabla u(x) = f'(r)\frac{x}{r}$.

4. Prove the basic lemma in the calculus of variations: Let $\Omega \subset \mathbb{R}^n$ be a domain and $f \in C(\Omega)$ such that
$$\int_\Omega f(x)h(x)\ dx = 0$$
   for all $h \in C_0^2(\Omega)$. Then $f \equiv 0$ in Ω.

5. Prove the basic lemma in the calculus of variations: Let $S = \partial\Omega$ be sufficiently regular and $f \in C^0(\partial\Omega)$ such that
$$\int_{\partial\Omega} f(x)h(x)\ dS = 0$$
   for all $h \in C(\partial\Omega)$. Then $f \equiv 0$ on $\partial\Omega$.

6. Write the minimal surface equation (1.1) as a quasilinear equation of second order.

7. Prove that a sufficiently regular minimizer in $C^1(\overline{\Omega})$ of
$$E(v) = \int_\Omega F(x, v, \nabla v)\ dx - \int_{\partial\Omega} g(v, v)\ ds,$$
is a solution of the boundary value problem
$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_{u_{x_i}} - F_u &= 0 \text{ in } \Omega \\ \sum_{i=1}^{n} F_{u_{x_i}} \nu_i - g_u &= 0 \text{ on } \partial\Omega, \end{aligned}$$
where $\nu = (\nu_1, \ldots, \nu_n)$ is the exterior unit nornal at the boundary $\partial\Omega$.

8. Prove that $\nu \cdot Tu = \cos\gamma$ on $\partial\Omega$, where γ is the angle between the container wall, which is here a cylinder, and the surface S defined by $u = u(x_1, x_2)$ at the boundary of S, ν is the exterior normal at $\partial\Omega$.

   Hint. The angle between two surfaces is by definition the angle between the two associated normals at the intersection of the surfaces.

9. Let $u \in C^2(\overline{\Omega})$ be a solution of
$$\begin{aligned} \mathrm{d}iv\ Tu &= C \text{ in } \Omega \\ \nu \cdot \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} &= \cos\gamma \text{ on } \partial\Omega, \end{aligned}$$
where C is a constant.

   Prove that
$$C = \frac{|\partial\Omega|}{|\Omega|} \cos\gamma\ .$$

   Hint. Integrate the differential equation over Ω.

10. Assume that $\Omega = B_R(0)$ is a disc with radius R and the center at the origin. Show that radially symmetric solutions $u(x) = w(r)$, $r = \sqrt{x_1^2 + x_2^2}$, of the capillary boundary value problem are solutions of
$$\begin{aligned} \left(\frac{rw'}{\sqrt{1 + w'^2}} \right)' &= \kappa rw \ \text{ in } 0 < r < R \\ \frac{w'}{\sqrt{1 + w'^2}} &= \cos\gamma \ \text{ if } r = R. \end{aligned}$$

Remark. It follows from a maximum principle of Concus and Finn [6] that a solution of the capillary equation over a disc must be radially symmetric.

11. Find all radially symmetric solutions of

$$\begin{aligned} \left(\frac{rw'}{\sqrt{1+w'^2}}\right)' &= Cr \ \text{ in } 0<r<R \\ \frac{w'}{\sqrt{1+w'^2}} &= \cos\gamma \ \text{ if } r=R. \end{aligned}$$

Hint. From an exercise above it follows that

$$C=\frac{2}{R}\cos\gamma.$$

12. Show that div Tu is twice the mean curvature of the surface defined by $z=u(x_1,x_2)$.

Chapter 2

Equations of first order

For a given sufficiently regular function F the general equation of first order for the unknown function $u(x)$ is

$$F(x, u, \nabla u) = 0$$

in $\Omega \in \mathbb{R}^n$. The main tool for studying related problems is the theory of ordinary differential equations. This is quite different for systems of partial differential of first order.

The general linear partial differential equation of first order can be written as

$$\sum_{i=1}^{n} a_i(x) u_{x_i} + c(x) u = f(x)$$

for given functions a_i, c and f. The general quasilinear partial differential equation of first order is

$$\sum_{i=1}^{n} a_i(x, u) u_{x_i} + c(x, u) = 0.$$

2.1 Linear equations

Let us begin with the linear homogeneous equation

$$a_1(x, y) u_x + a_2(x, y) u_y = 0. \tag{2.1}$$

Assume there is a C^1-solution $z = u(x, y)$. This function defines a surface S which has at $P = (x, y, u(x, y))$ the normal

$$\mathbf{N} = \frac{1}{\sqrt{1 + |\nabla u|^2}}(-u_x, -u_y, 1)$$

and the tangential plane

$$\zeta - z = u_x(x, y)(\xi - x) + u_y(x, y)(\eta - y).$$

Set $p = u_x(x, y)$, $q = u_y(x, y)$ and $z = u(x, y)$. The tupel (x, y, z, p, q) is called *surface element* and the tupel (x, y, z) *support* of the surface element. The tangential plane is defined by the surface element. On the other hand, differential equation (2.1)

$$a_1(x, y)p + a_2(x, y)q = 0$$

defines at each support (x, y, z) a bundle of planes if we consider all (p, q) satisfying this equation. For fixed (x, y) this family of planes $\Pi(\lambda) = \Pi(\lambda; x, y)$ is defined by a one parameter family of ascents $p(\lambda) = p(\lambda; x, y)$, $q(\lambda) = q(\lambda; x, y)$. The envelope of these planes is a line since

$$a_1(x, y)p(\lambda) + a_2(x, y)q(\lambda) = 0,$$

which implies that the normal $\mathbf{N}(\lambda)$ on $\Pi(\lambda)$ is perpendicular on $(a_1, a_2, 0)$.

Consider a curve $\mathbf{x}(\tau) = (x(\tau), y(\tau), z(\tau))$ on $\mathcal{S}$, let $T_{\mathbf{x}_0}$ be the tangential plane at $\mathbf{x}_0 = (x(\tau_0), y(\tau_0), z(\tau_0))$ of $\mathcal{S}$ and consider the line on $T_{\mathbf{x}_0}$

$$L: \quad l(\sigma) = \mathbf{x}_0 + \sigma \mathbf{x}'(\tau_0), \quad \sigma \in \mathbb{R},$$

see Figure 2.1

We assume L coincides with the envelope, which is a line here, of the family of planes $\Pi(\lambda)$ at (x, y, z). Assume that $T_{\mathbf{x}_0} = \Pi(\lambda_0)$ and consider two planes

$$\begin{array}{rcl} \Pi(\lambda_0): \quad z - z_0 & = & (x - x_0)p(\lambda_0) + (y - y_0)q(\lambda_0) \\ \Pi(\lambda_0 + h): \quad z - z_0 & = & (x - x_0)p(\lambda_0 + h) + (y - y_0)q(\lambda_0 + h). \end{array}$$

At the intersection $l(\sigma)$ we have

$$(x - x_0)p(\lambda_0) + (y - y_0)q(\lambda_0) = (x - x_0)p(\lambda_0 + h) + (y - y_0)q(\lambda_0 + h).$$

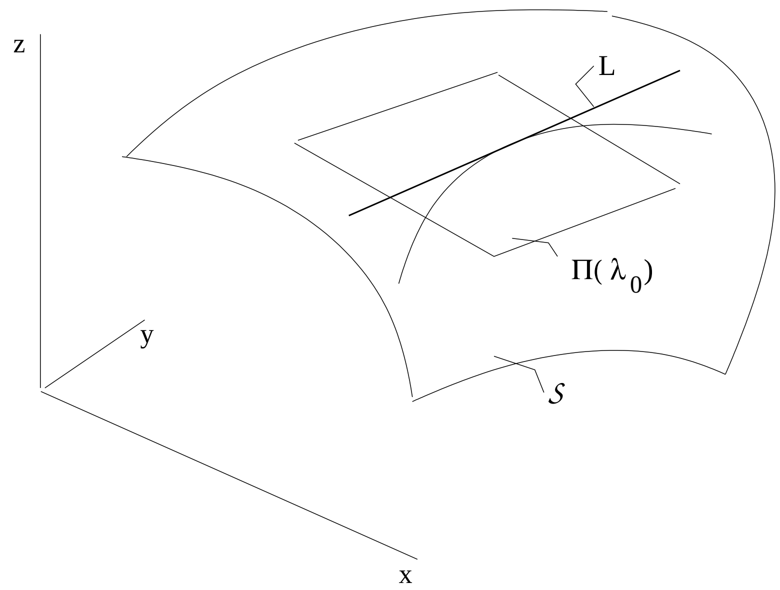

Figure 2.1: Curve on a surface

Thus,

$$x'(\tau_0)p'(\lambda_0) + y'(\tau_0)q'(\lambda_0) = 0.$$

From the differential equation

$$a_1(x(\tau_0), y(\tau_0))p(\lambda) + a_2(x(\tau_0), y(\tau_0))q(\lambda) = 0$$

it follows

$$a_1 p'(\lambda_0) + a_2 q'(\lambda_0) = 0.$$

Consequently

$$(x'(\tau), y'(\tau)) = \frac{x'(\tau)}{a_1(x(\tau, y(\tau))}(a_1(x(\tau), y(\tau)), b(x(\tau), y(\tau)),$$

since τ_0 was an arbitrary parameter. Here we assume that $x'(\tau) \neq 0$ and $a_1(x(\tau), y(\tau)) \neq 0$.

Then we introduce a new parameter t by the inverse of $\tau = \tau(t)$, where

$$t(\tau) = \int_{\tau_0}^{\tau} \frac{x'(s)}{a_1(x(s), y(s))}\ ds.$$

It follows $x'(t) = a_1(x, y),\ y'(t) = a_2(x, y)$. We denote $\mathbf{x}(\tau(t))$ again by $\mathbf{x}(t)$.

Now, we consider the initial value problem

$$x'(t) = a_1(x, y), \quad y'(t) = a_2(x, y), \quad x(0) = x_0, \ y(0) = y_0. \tag{2.2}$$

From the theory of ordinary differential equations it follows (Theorem of Picard-Lindelöf) that there is a unique solution in a neighbouhood of $t = 0$ provided the functions $a_1, \ a_2$ are in C^1. From this definition of the curves $(x(t), y(t))$ is follows that the field of directions $(a_1(x_0, y_0), a_2(x_0, y_0))$ defines the slope of these curves at $(x(0), y(0))$.

Definition. Differential equations in (2.2) are called *characteristic equations* or characteristic system and solutions of the associated initial value problem are called *characteristic curves*.

Definition. A function $\phi(x, y)$ is said to be an *integral* of the characteristic system if $\phi(x(t), y(t)) = const.$ for each characteristic curve. The constant depends on the characteristic curve considered.

Proposition 2.1. *Assume $\phi \in C^1$ is an integral, then $u = \phi(x, y)$ is a solution of (2.1).*

Proof. Consider for given (x_0, y_0) the above initial value problem (2.2). Since $\phi(x(t), y(t)) = const.$ it follows

$$\phi_x x' + \phi_y y' = 0$$

for $|t| < t_0$, $t_0 > 0$ and sufficiently small. Thus

$$\phi_x(x_0, y_0)a_1(x_0, y_0) + \phi_y(x_0, y_0)a_2(x_0, y_0) = 0.$$

Remark. If $\phi(x, y)$ is a solution of equation (2.1) then also $H(\phi(x, y))$, where $H(s)$ is a given C^1-function.

Examples

1. Consider

$$a_1 u_x + a_2 u_y = 0,$$

where $a_1,\ a_2$ are constants. The system of characteristic equations is

$$x' = a_1,\ y' = a_2.$$

Thus, the characteristic curves are parallel straight lines defined by

$$x = a_1 t + A,\ y = a_2 t + B,$$

where $A,\ B$ are arbitrary constants. From these equations it follow that

$$\phi(x,y) := a_2 x - a_1 y$$

is constant along each characteristic curve. Consequently, see Proposition 2.1, $u = a_2 x - a_1 y$ is a solution of the differential equation. From an exercise it follows that

$$u = H(a_2 x - a_1 y), \tag{2.3}$$

where $H(s)$ is an arbitrary C^1-function, is also a solution. Since u is constant when $a_2 x - a_1 y$ is constant, equation (2.3) defines cylinder surfaces which are generated by parallel straight lines which are parallel to the (x,y)-plane, see Figure 2.2.

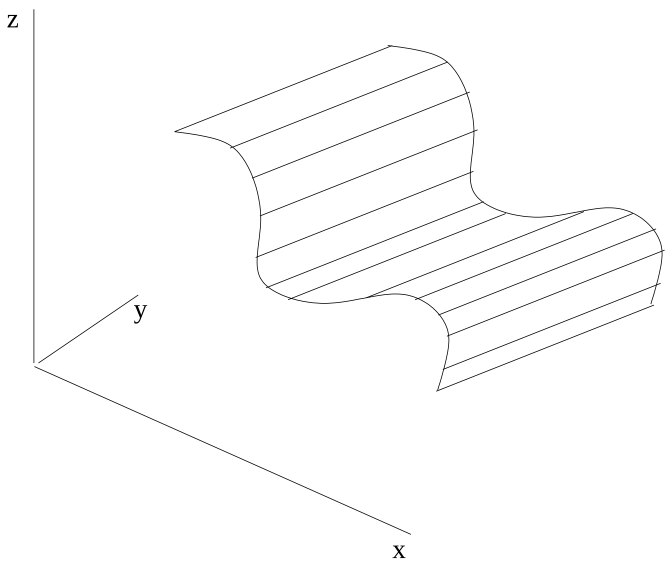

Figure 2.2: Cylinder surfaces

2. Consider the differential equation

$$xu_x + yu_y = 0.$$

The characteristic equations are

$$x' = x, \; y' = y,$$

and the characteristic curves are given by

$$x = A\mathrm{e}^t, \; y = B\mathrm{e}^t,$$

where $A, \; B$ are arbitrary constants. Thus, an integral is y/x, $x \neq 0$, and for a given C^1-function the function $u = H(x/y)$ is a solution of the differential equation. If $y/x = const.$, then u is constant. Suppose that $H'(s) > 0$, for example, then u defines right helicoids (german: Wendelflächen), see Figure 2.3

Figure 2.3: Right helicoid (Museo Ideale Leonardo da Vinci, Italy)

3. Consider the differential equation

$$yu_x - xu_y = 0.$$

The associated characteristic system is

$$x' = y, \ y' = -x.$$

If follows

$$x'x + yy' = 0,$$

or, equivalently,

$$\frac{d}{dt}(x^2 + y^2) = 0,$$

which implies that $x^2 + y^2 = const.$ along each characteristic. Thus, rotationally symmetric surfaces defined by $u = H(x^2 + y^2)$, where $H' \neq 0$, are solutions of the differential equation.

4. The associated characteristic equations to

$$ayu_x + bxu_y = 0,$$

where $a, \ b$ are positive constants, are given by

$$x' = ay, \ y' = bx.$$

It follows $bxx' - ayy' = 0$, or equivalently,

$$\frac{d}{dt}(bx^2 - ay^2) = 0.$$

Thus, solutions of the differential equation are $u = H(bx^2 - ay^2)$, which define surfaces which have a hyperbola as the intersection with planes parallel to the (x, y)-plane. Here is $H(s)$ an arbitrary C^1-function.

2.2 Quasilinear equations

Here we consider equation

$$a_1(x, y, u)u_x + a_2(x, y, u)u_y = a_3(x, y, u). \tag{2.4}$$

The inhomogeneous linear equation

$$a_1(x, y)u_x + a_2(x, y)u_y = a_3(x, y)$$

is a special case of (2.4).

One arrives at characteristic equations $x' = a_1,\ y' = a_2,\ z' = a_3$ from (2.4) by the same arguments as in the case of homogeneous linear equations in two variables. The additional equation $z' = a_3$ follows from

$$\begin{aligned} z'(\tau) &= p(\lambda)x'(\tau) + q(\lambda)y'(\tau) \\ &= pa_1 + qa_2 \\ &= a_3, \end{aligned}$$

see also Section 2.3, where the general case of nonlinear equations in two variables is considered.

2.2.1 A linearization method

We can transform the inhomogeneous equation (2.4) into a homogeneous linear equation for an unknown function of three variables by the following trick.

We are looking for a function $\psi(x, y, u)$ such that the solution $u = u(x, y)$ of (2.4) is defined implicitely by $\psi(x, y, u) = const$. Assume there is such a function ψ und let u be a solution of (2.4), then

$$\psi_x + \psi_u u_x = 0, \quad \psi_y + \psi_u u_y = 0.$$

Assume $\psi_u \neq 0$, then

$$u_x = -\frac{\psi_x}{\psi_u}, \quad u_y = -\frac{\psi_y}{\psi_u}.$$

Then, it follows from (2.4) the linear homogeneous equation

$$a_1(x, y, z)\psi_x + a_2(x, y, z)\psi_y + a_3(x, y, z)\psi_z = 0, \tag{2.5}$$

where $z := u$.

We consider the associated system of characteristic equations

$$\begin{aligned} x'(t) &= a_1(x, y, z) \\ y'(t) &= a_2(x, y, z) \\ z'(t) &= a_3(x, y, z). \end{aligned}$$

One arrives at this system by the same arguments as in the two dimensional case above.

Proposition 2.2. (i) *Assume $w \in C^1$, $w = w(x, y, z)$, is an integral, that is, it is constant along each fixed solution of (2.5), then $\psi = w(x, y, z)$ is a solution of (2.5).*
(ii) *The function $z = u(x, y)$, implicitely defined through $\psi(x, u, z) = const.$, is a solution of (2.4), provided that $\psi_z \neq 0$.*
(iii) *Let $z = u(x, y)$ be a solution of (2.4) and let $(x(t), y(t))$ be a solution of*

$$x'(t) = a_1(x, y, u(x, y)), \quad y'(t) = a_2(x, y, u(x, y)),$$

then $z(t) := u(x(t), y(t))$ satisfies the third of the above characteristic equations.

Proof. Exercise.

2.2.2 Initial value problem of Cauchy

Consider again the quasilinear equation

$$(\star) \qquad a_1(x, y, u)u_x + a_2(x, y, u)u_y = a_3(x, y, u).$$

Let

$$\Gamma : \quad x = x_0(s),\ y = y_0(s),\ z = z_0(s),\ s_1 \leq s \leq s_2,\ -\infty < s_1 < s_2 < +\infty,$$

be a regular curve in $\mathbb{R}^3$ and denote by $\mathcal{C}$ the orthogonal projection of Γ onto the (x, y)-plane, that is,

$$\mathcal{C} : \quad x = x_0(s), \quad y = y_0(s).$$

Initial value problem of Cauchy: *Find a C^1-solution $u = u(x, y)$ of $(\star)$ such that $u(x_0(s), y_0(s)) = z_0(s)$, that is, we seek a surface $\mathcal{S}$ defined by $z = u(x, y)$ which contains the curve Γ.*

Definition. The curve Γ is said to be *noncharacteristic* if

$$x_0'(s)a_2(x_0(s), y_0(s)) - y_0'(s)a_1(x_0(s), y_0(s)) \neq 0.$$

Remark. If $x_0(s)$, $y_0(s)$, $z_0(s)$ is a solution of the characteristic system, then Γ is not noncharacteristic, it is by definition a characteristic curve.

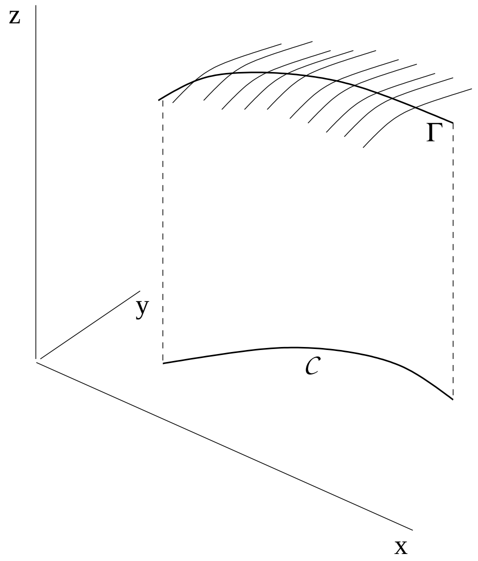

Figure 2.4: Cauchy initial value problem

Theorem 2.1. *Assume $a_1,\ a_2,\ a_2 \in C^1$ in their arguments, the initial data $x_0,\ y_0,\ z_0 \in C^1[s_1, s_2]$ and Γ is noncharacteristic.*

Then there is a neighbourhood of $\mathcal{C}$ such that there exists exactly one solution u of the Cauchy initial value problem.

Proof. (i) Existence. Consider the following initial value problem for the system of characteristic equations to $(\star)$:

$$\begin{aligned} x'(t) &= a_1(x, y, z) \\ y'(t) &= a_2(x, y, z) \\ z'(t) &= a_3(x, y, z) \end{aligned}$$

with the initial conditions

$$\begin{aligned} x(s, 0) &= x_0(s) \\ y(s, 0) &= y_0(s) \\ z(s, 0) &= z_0(s). \end{aligned}$$

Let $x = x(s, t)$, $y = y(s, t)$, $z = z(s, t)$ be the solution, $s_1 \le s \le s_2$, $|t| < \eta$ for an $\eta > 0$. We will show that this set of strings sticked onto the curce Γ, see Figure 2.4, defines a surface. To show this, we consider the inverse functions $s = s(x, y)$, $t = t(x, y)$ of $x = x(s, t)$, $y = y(s, t)$ and show that $z(s(x, y), t(x, y))$ is a solution of the initial problem of Cauchy. The inverse functions s and t exist in a neighbourhood of $t = 0$ since

$$\det \frac{\partial(x, y)}{\partial(s, t)}\Big|_{t=0} = \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix}_{t=0} = x_0'(s)a_2 - y_0'(s)a_1 \neq 0$$

since the initial curve Γ is noncharacteristic by assumption.

Set

$$u(x, y) := z(s(x, y), t(x, y)),$$

then u satisfies the initial condition since

$$u(x, y)|_{t=0} = z(s, 0) = z_0(s).$$

The following calculation shows that u is also a solution of the differential equation ($\star$).

$$\begin{aligned} a_1 u_x + a_2 u_y &= a_1(z_s s_x + z_t t_x) + a_2(z_s s_y + z_t t_y) \\ &= z_s(a_1 s_x + a_2 s_y) + z_t(a_1 t_x + a_2 t_y) \\ &= z_s(s_x x_t + s_y y_t) + z_t(t_x x_t + t_y y_t) \\ &= a_3 \end{aligned}$$

since $0 = s_t = s_x x_t + s_y y_t$ and $1 = t_t = t_x x_t + t_y y_t$.

(ii) Uniqueness. Suppose that $v(x, y)$ is a second solution. Consider a point (x', y') in a neighbourhood of the curve $(x_0(s), y(s))$, $s_1 + \epsilon \leq s \leq s_2 - \epsilon$, $\epsilon > 0$ small. The inverse parameters, see above, are $s' = s(x', y')$, $t' = t(x', y')$, see Figure 2.5.

Let

$$\mathcal{A}: \quad x(t) := x(s', t),\ y(t) := y(s', t),\ z(t) := z(s', t)$$

be the solution of the above initial value problem for the characteristic differential equations with the initial data

$$x(s', 0) = x_0(s'),\ y(s', 0) = y_0(s'),\ z(s', 0) = z_0(s').$$

According to the above construction this curve is on the surface $\mathcal{S}$ defined by $u = u(x, y)$ and $u(x', y') = z(s', t')$. Set

$$\psi(t) := v(x(t), y(t)) - z(t),$$

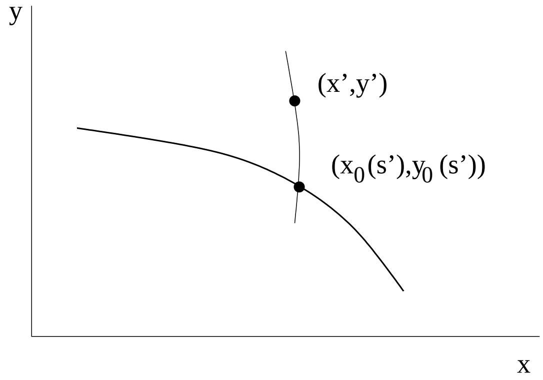

Figure 2.5: Uniqueness proof

then

$$\begin{aligned}\psi'(t) &= v_x x' + v_y y' - z'\\ &= x_x a_1 + v_y a_2 - a_3 = 0\end{aligned}$$

since v is a solution of the differential equation by assumption. Since v satisfies the initial condition one has

$$\psi(0) = v(x(s',0), y(s',0)) - z(s',0) = 0.$$

Thus, $\psi(t) \equiv 0$, that is,

$$v(x(s',t), y(s',t)) - z(s',t) = 0.$$

Set $t = t'$, then

$$v(x',y') - z(s',t') = 0,$$

which shows that $v(x',y') = u(x',y')$ since $z(s',t') = u(x',y')$. $\square$

Remark. In general, there is no uniqueness if the initial curve Γ is a characteristic curve, see Figure 2.6 and an exercise.

Examples

1. Consider the Cauchy initial value problem

$$u_x + u_y = 0$$

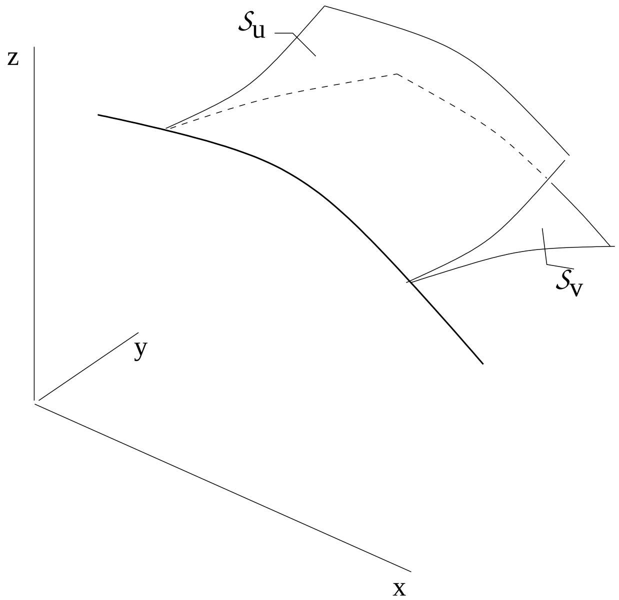

Figure 2.6: Multiple solutions

with the initial data

$$x_0(s) = s,\ y_0(s) = 1,\ z_0(s) \text{ is a given } C^1\text{-function.}$$

These initial data are noncharacteristic since $y_0'a_1 - x_0'a_2 = -1$. The solution of the associated system of characteristic equations

$$x'(t) = 1,\ y'(t) = 1, u'(t) = 0$$

with the initial conditions

$$x(s,0) = x_0(s),\ y(s,0) = y_0(s),\ z(s,0) = z_0(s)$$

is given by

$$x = t + x_0(s),\ y = t + y_0(s),\ z = z_0(s),$$

that is,

$$x = t + s,\ y = t + 1,\ z = z_0(s).$$

It follows $s = x - y + 1,\ t = y - 1$ and that $u = z_0(x - y + 1)$ is the solution of the Cauchy initial value problem.

2. A problem from kinetics in chemistry. Consider for $x \geq 0$, $y \geq 0$ the problem

$$u_x + u_y = \left(k_0 \mathrm{e}^{-k_1 x} + k_2\right)(1-u)$$

with initial data

$$u(x,0) = 0,\ x > 0,\ \text{and}\ u(0,y) = u_0(y),\ y > 0.$$

Here the constants k_j are positive, these constants define the velocity of the reactions in consideration, and the function $u_0(y)$ is given. The variable x is the time and y is the hight of a tube, for example, in which the chemical reaction takes place, and u is the concentration of the chemical substance.

In contrast to our previous assumptions, the initial data are not in C^1. The projection $\mathcal{C}_1 \cup \mathcal{C}_2$ of the initial curve onto the (x,y)-plane has a corner at the origin, see Figure 2.7.

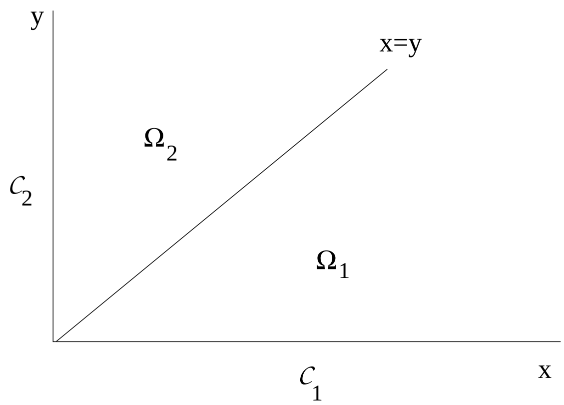

Figure 2.7: Domains to the chemical kinetics example

The associated system of characteristic equations is

$$x'(t) = 1,\ y'(t) = 1,\ z'(t) = \left(k_0 \mathrm{e}^{-k_1 x} + k_2\right)(1-z).$$

It follows $x = t + c_1$, $y = t + c_2$ with constants c_j. Thus, the projection of the characteristic curves on the (x,y)-plane are straight lines parallel to $y = x$. We will solve the initial value problems in the domains Ω_1 and Ω_2, see Figure 2.7, separately.

(i) The initial value problem in Ω_1. The initial data are

$$x_0(s) = s,\ y_0(s) = 0,\ z_0(0) = 0,\ s \geq 0.$$

It follows

$$x = x(s,t) = t + s,\ \ y = y(s,t) = t.$$

Thus,

$$z'(t) = (k_0 \mathrm{e}^{-k_1(t+s)} + k_2)(1 - z),\ \ z(0) = 0.$$

The solution of this initial value problem is given by

$$z(s,t) = 1 - \exp\left(\frac{k_0}{k_1}\mathrm{e}^{-k_1(s+t)} - k_2 t - \frac{k_0}{k_1}\mathrm{e}^{-k_1 s}\right).$$

Consequently,

$$u_1(x,y) = 1 - \exp\left(\frac{k_0}{k_1}\mathrm{e}^{-k_1 x} - k_2 y - k_0 k_1 \mathrm{e}^{-k_1(x-y)}\right)$$

is the solution of the Cauchy initial value problem in Ω_1. If time x tends to ∞, we get the limit

$$\lim_{x\to\infty} u_1(x,y) = 1 - \mathrm{e}^{-k_2 y}.$$

(ii) The initial value problem in Ω_2. The initial data are here

$$x_0(s) = 0,\ \ y_0(s) = s,\ \ z_0(0) = u_0(s),\ \ s \geq 0.$$

It follows

$$x = x(s,t) = t,\ \ y = y(s,t) = t + s.$$

Thus,

$$z'(t) = (k_0 \mathrm{e}^{-k_1 t} + k_2)(1 - z),\ \ z(0) = 0.$$

The solution of this initial value problem is given by

$$z(s,t) = 1 - (1 - u_0(s))\exp\left(\frac{k_0}{k_1}\mathrm{e}^{-k_1 t} - k_2 t - \frac{k_0}{k_1}\right).$$

Consequently,

$$u_2(x,y) = 1 - (1 - u_0(y-x))\exp\left(\frac{k_0}{k_1}\mathrm{e}^{-k_1 x} - k_2 x - \frac{k_0}{k_1}\right)$$

is the solution in Ω_2.

If $x = y$, then

$$\begin{aligned} u_1(x,y) &= 1 - \exp\left(\frac{k_0}{k_1}\mathrm{e}^{-k_1 x} - k_2 x - \frac{k_0}{k_1}\right) \\ u_2(x,y) &= 1 - (1 - u_0(0))\exp\left(\frac{k_0}{k_1}\mathrm{e}^{-k_1 x} - k_2 x - \frac{k_0}{k_1}\right). \end{aligned}$$

If $u_0(0) > 0$, then $u_1 < u_2$ if $x = y$, that is, there is a jump of the concentration of the substrate along its burning front defined by $x = y$.

The case if a solution of the equation is known

Here we will see that we get immediately a solution of the Cauchy initial value problem if a solution of the *homogeneous linear equation*

$$a_1(x,y)u_x + a_2(x,y)u_y = 0$$

is known.

Let

$$x_0(s),\ y_0(s),\ z_0(s),\ s_1 < s < s_2$$

be the initial data and let $u = \phi(x,y)$ be a solution of the differential equation. We assume that

$$\phi_x(x_0(s), y_0(s))x_0'(s) + \phi_y(x_0(s), y_0(s))y_0'(s) \neq 0$$

is satisfied. Set $g(s) = \phi(x_0(s), y_0(s))$ and let $s = h(g)$ be the inverse function.

The solution of the Cauchy initial problem is given by $u_0\left(h(\phi(x,y))\right)$.

This follows since a composition of a solution is a solution again, see an exercise, and since

$$u_0\left(h(\phi(x_0(s), y_0(s)))\right) = u_0(h(g)) = u_0(s).$$

Example: Consider equation

$$u_x + u_y = 0$$

with initial data

$$x_0(s) = s,\ y_0(s) = 1,\ u_0(s) \text{ is a given function.}$$

A solution of the differential equation is $\phi(x,y) = x - y$. Thus, $\phi((x_0(s), y_0(s)) = s - 1$ and $u_0(\phi + 1) = u_0(x - y + 1)$ is the solution of the problem.

2.3 Nonlinear equations in two variables

Here we consider equation

$$F(x, y, z, p, q) = 0, \tag{2.6}$$

where $z = u(x, y)$, $p = u_x(x, y)$, $q = u_y(x, y)$ and $F \in C^2$ is given such that $F_p^2 + F_q^2 \neq 0$.

In contrast to the quasilinear case, this general nonlinear equation is more complicated. Together with (2.6) we will consider the following system of ordinary equations which follow from considerations below as necessary conditions, in particular from the assumption that there is a solution of (2.6).

$$\begin{aligned}
x'(t) &= F_p & (2.7)\\
y'(t) &= F_q & (2.8)\\
z'(t) &= pF_p + qF_q & (2.9)\\
p'(t) &= -F_x - F_u p & (2.10)\\
q'(t) &= -F_y - F_u q. & (2.11)
\end{aligned}$$

Definition. Equations (2.7)–(2.11) are said to be *characteristic equations* of equation (2.6) and a solution

$$(x(t), y(t), z(t), p(t), q(t))$$

of the characteristic equations is called *characteristic strip* or *Monge curve.*

We will see, as in the quasilinear case, that the strips defined by the characteristic equations build the solution surface of the Cauchy initial value problem.

Let $z = u(x, y)$ be a solution of the general nonlinear differential equation (2.6).

Let (x_0, y_0, z_0) be fixed, then equation (2.6) defines a set of planes given by (x_0, y_0, z_0, p, q), that is, planes given by $z = v(x, y)$ which contain the point (x_0, y_0, z_0) and for which $v_x = p$, $v_y = q$ at (x_0, y_0). In the case of quasilinear equations these set of planes is a boundle of planes which all contain a fixed straight line, see Section 2.1. In the general case of this section the situation is more complicated.

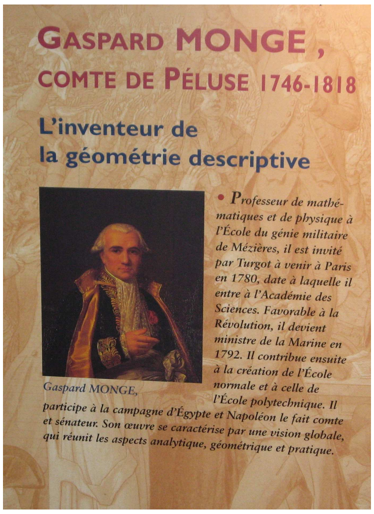

Figure 2.8: Gaspard Monge (Panthéon, Paris)

Consider example

$$p^2 + q^2 = f(x, y, z), \tag{2.12}$$

where f is a given positive function. Let E be a plane defined by $z = v(x, y)$ and which contains (x_0, y_0, z_0). Then the normal on the plane E directed downward is

$$\mathbf{N} = \frac{1}{\sqrt{1 + |\nabla v|^2}}(p, q, -1),$$

where $p = v_x(x_0, y_0)$, $q = v_y(x_0, y_0)$. It follows from (2.12) that the normal $\mathbf{N}$ makes a constant angle with the z-axis, and the z-coordinate of $\mathbf{N}$ is constant, see Figure 2.9.

Thus, the endpoints of the normals, fixed at (x_0, y_0, z_0), define a circle parallel to the (x, y)-plane, that is, there is a cone which is the envelope of all these planes.

We assume that the general equation (2.6) defines such a Monge cone at each point in $\mathbb{R}^3$. Then we seek a surface S which touches at each point its Monge cone, see Figure 2.10.

More precisely, we assume there exists, as in the above example, a one parameter C^1-family

$$p(\lambda) = p(\lambda; x, y, z), \;\; q(\lambda) = q(\lambda; x, y, z)$$

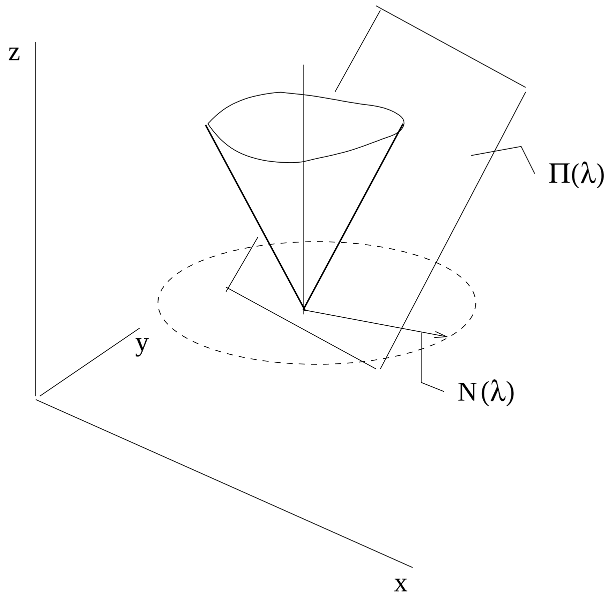

Figure 2.9: Monge cone in an example

of solutions of (2.6). These $(p(\lambda), q(\lambda))$ define a family $\Pi(\lambda)$ of planes.

Let

$$\mathbf{x}(\tau) = (x(\tau), y(\tau), z(\tau))$$

be a curve on the surface S which touches at each point its Monge cone, see Figure 2.11. That is, we assume that at each point of the surface $\mathcal{S}$ the associated tangent plane coincides with a plane from the family $\Pi(\lambda)$ at this point.

Consider the tangential plane $T_{\mathbf{x}_0}$ of the surface S at $\mathbf{x}_0 = (x(\tau_0), y(\tau_0), z(\tau_0))$. The straight line

$$\mathbf{l}(\sigma) = \mathbf{x}_0 + \sigma \mathbf{x}'(\tau_0), \qquad -\infty < \sigma < \infty,$$

is an apothem (german: Mantellinie) of the cone by assumption and is contained in the tangential plane $T_{\mathbf{x}_0}$ as the tangent of a curve on the surface S. It is

$$\mathbf{x}'(\tau_0) = \mathbf{l}'(\sigma). \tag{2.13}$$

The straight line $\mathbf{l}(\sigma)$ satisfies

$$l_3(\sigma) - z_0 = (l_1(\sigma) - x_0)p(\lambda_0) + (l_2(\sigma) - y_0)q(\lambda_0),$$

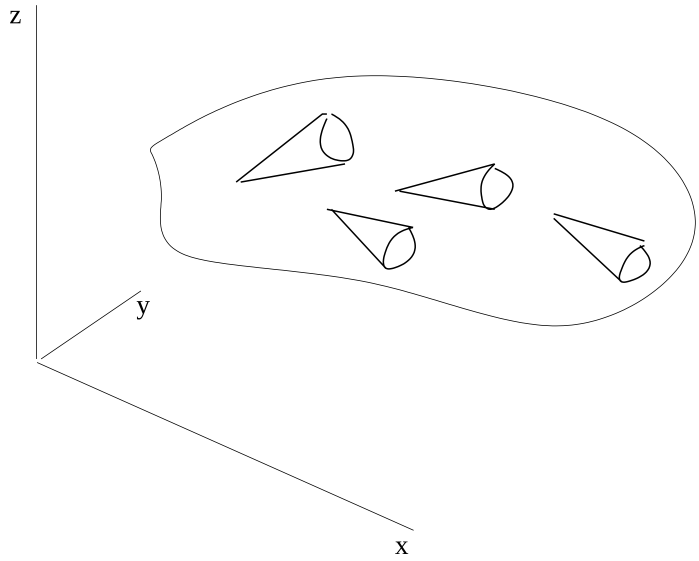

Figure 2.10: Monge cones

since it is contained in the tangential plane $T_{\mathbf{x}_0}$ defined by the slope (p, q). It follows

$$l_3'(\sigma) = p(\lambda_0)l_1'(\sigma) + q(\lambda_0)l_2'(\sigma).$$

Together with (2.13) we obtain

$$z'(\tau) = p(\lambda_0)x'(\tau) + q(\lambda_0)y'(\tau). \tag{2.14}$$

The above straight line $\mathbf{l}$ is the limit of the intesection line of two neighbouring planes which envelopes the Monge cone:

$$\begin{aligned} z - z_0 &= (x - x_0)p(\lambda_0) + (y - y_0)q(\lambda_0) \\ z - z_0 &= (x - x_0)p(\lambda_0 + h) + (y - y_0)q(\lambda_0 + h). \end{aligned}$$

On the intersection one has

$$(x - x_0)p(\lambda) + (y - y_0)q(\lambda_0) = (x - x_0)p(\lambda_0 + h) + (y - y_0)q(\lambda_0 + h).$$

Let $h \to 0$, it follows

$$(x - x_0)p'(\lambda_0) + (y - y_0)q'(\lambda_0) = 0.$$

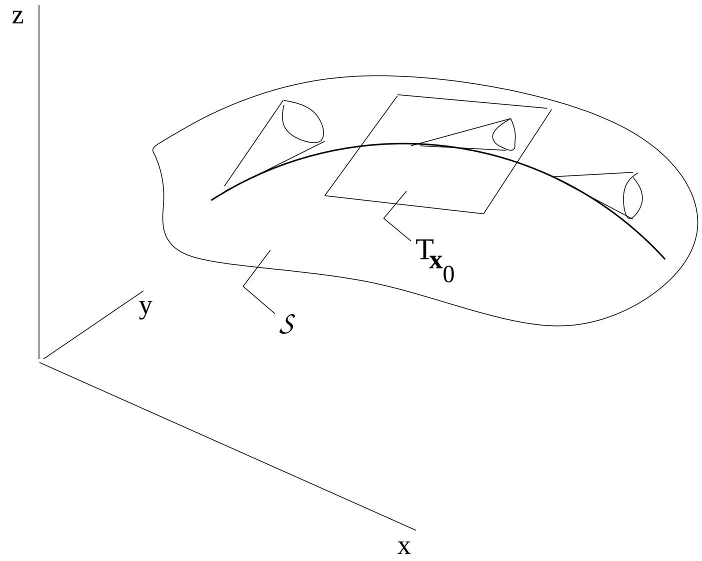

Figure 2.11: Monge cones along a curve on the surface

Since $x = l_1(\sigma)$, $y = l_2(\sigma)$ in this limit position, we have

$$p'(\lambda_0)l_1'(\sigma) + q'(\lambda_0)l_2'(\sigma) = 0,$$

and it follows from (2.13) that

$$p'(\lambda_0)x'(\tau) + q'(\lambda_0)y'(\tau) = 0. \tag{2.15}$$

From differential equation $F(x_0, y_0, z_0, p(\lambda), q(\lambda)) = 0$ we see that

$$F_p p'(\lambda) + F_q q'(\lambda) = 0. \tag{2.16}$$

Assume $x'(\tau_0) \neq 0$ and $F_p \neq 0$, then we obtain from (2.15), (2.16)

$$\frac{y'(\tau_0)}{x'(\tau_0)} = \frac{F_q}{F_p},$$

and from (2.14) (2.16) that

$$\frac{z'(\tau_0)}{x'(\tau_0)} = p + q\frac{F_q}{F_p}.$$

It follows, since τ_0 was an arbitrary fixed parameter,

$$\begin{aligned}
\mathbf{x}'(\tau) &= (x'(\tau), y'(\tau), z'(\tau)) \\
&= \left(x'(\tau), x'(\tau)\frac{F_q}{F_p}, x'(\tau)\left(p + q\frac{F_q}{F_p}\right)\right) \\
&= \frac{x'(\tau)}{F_p}(F_p, F_q, pF_p + qF_q).
\end{aligned}$$

That is, the tangential vector $\mathbf{x}'(\tau)$ is proportional to (F_p, F_q, pF_p+qF_q). Set

$$a(\tau) = \frac{x'(\tau)}{F_p},$$

where $F = F(x(\tau), y(\tau), z(\tau), p(\lambda(\tau)), q(\lambda(\tau)))$. Introducing the new parameter t by the inverse of $\tau = \tau(t)$, where

$$t(\tau) = \int_{\tau_0}^{\tau} a(s)\ ds,$$

we obtain the characteristic equations (2.7)–(2.9). Here we denote $\mathbf{x}(\tau(t))$ by $\mathbf{x}(t)$ again. From the differential equation (2.6) and (2.7)–(2.9) we obtain equations (2.10) and (2.11). Assume that the surface $z = u(x, y)$ under consideration belongs to the class C^2, then

$$\begin{aligned}
F_x + F_z p + F_p p_x + F_q p_y &= 0, \quad (q_x = p_y) \\
F_x + F_z p + x'(t)p_x + y'(t)p_y &= 0 \\
F_x + F_z p + p'(t) &= 0
\end{aligned}$$

since $p = p(x, y) = p(x(t), y(t))$ on the curve $\mathbf{x}(t)$. Thus equation (2.10) of the characteristic system is shown. Differentiating the differential equation(2.6) with respect to y, we get finally equation (2.11).

Remark. In the previous quasilinear case

$$F(x, y, z, p, q) = a_1(x, y, z)p + a_2(x, y, z)q - a_3(x, y, z)$$

the first three characteristic equations are the same:

$$x'(t) = a_1(x, y, z),\ \ y'(t) = a_2(x, y, z),\ \ z'(t) = a_3(x, y, z).$$

The point is that the right hand sides are independent on p or q. It follows from Theorem 2.1 that there exists a solution of the Cauchy initial value problem provided the initial data are noncharacteristic. That is, we do not need the other remaining two characteristic equations.

The other two equations (2.10) and (2.11) are satisfied in this quasilinear case automatically if there is a solution of the equation, see the above derivation of these equations.

The geometric meaning of the first three characteristic differential equations (2.7)–(2.11) is the following one. Each point of the curve $\mathcal{A}: \quad (x(t), y(t), z(t))$ corresponds a tangential plane with the normal direction $(-p, -q, 1)$ such that

$$z'(t) = p(t)x'(t) + q(t)y'(t).$$

This equation is called *strip condition*. On the other hand, let $z = u(x, y)$ define a surface, then $z(t) := u(x(t), y(t))$ satisfies the strip condition, where $p = u_x$ and $q = u_y$, that is, the "scales" defined by the normals fit together.

Proposition 2.3. *$F(x, y, z, p, q)$ is an integral, that is, it is constant along each characteristic curve.*

Proof.

$$\begin{aligned}
\frac{d}{dt}F(x(t), y(t), z(t), p(t), q(t)) &= F_x x' + F_y y' + F_z z' + F_p p' + F_q q' \\
&= F_x F_p + F_y F_q + pF_z F_p + qF_z F_q \\
&\quad -F_p f_x - F_p F_z p - F_q F_y - F_q F_z q \\
&= 0.
\end{aligned}$$

Corollary. Assume $F(x_0, y_0, z_0, p_0, q_0) = 0$, then $F = 0$ along characteristc curves with the initial data $(x_0, y_0, z_0, p_0, q_0)$.

Proposition 2.4. *Let $z = u(x, y)$, $u \in C^2$, be a solution of the nonlinear equation (2.6). Set*

$$z_0 = u(x_0, y_0,)\ p_0 = u_x(x_0, y_0),\ \ q_0 = u_y(x_0, y_0).$$

Then the associated characteristic strip is in the surface $\mathcal{S}$ defined by $z = u(x, y)$. That is,

$$\begin{aligned} z(t) &= u(x(t), y(t)) \\ p(t) &= u_x(x(t), y(t)) \\ q(t) &= u_y(x(t), y(t)), \end{aligned}$$

where $(x(t), y(t), z(t), p(t), q(t))$ is the solution of the characteristic system (2.7)–(2.11) with initial data $(x_0, y_0, z_0, p_0, q_0)$

Proof. Consider the initial value problem

$$\begin{aligned} x'(t) &= F_p(x, y, u(x, y), u_x(x, y), u_y(x, y)) \\ y'(t) &= F_q(x, y, u(x, y), u_x(x, y), u_y(x, y)) \end{aligned}$$

with the initial data $x(0) = x_0$, $y(0) = y_0$. We will show that

$$(x(t), y(t), u(x(t), y(t)), u_x(x(t), y(t)), u_y(x(t), y(t)))$$

is a solution of the characteristic system. We recall that the solution exists and is *uniquely determined*.

Set $z(t) = u(x(t), y(t))$, then $(x(t), y(t), z(t)) \subset \mathcal{S}$, and

$$z'(t) = u_x x'(t) + u_y y'(t) = u_x F_p + u_y F_q.$$

Set $p(t) = u_x(x(t), y(t))$, $q(t) = u_y(x(t), y(t))$, then

$$\begin{aligned} p'(t) &= u_{xx} F_p + u_{xy} F_q \\ q'(t) &= u_{yx} F_p + u_{yy} F_q. \end{aligned}$$

Finally, from differential equation $F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0$ it follows

$$\begin{aligned} p'(t) &= -F_x - F_u p \\ q'(t) &= -F_y - F_u q. \end{aligned}$$

$\square$

2.3.1 Initial value problem of Cauchy

Let

$$x = x_0(s),\ y = y_0(s),\ z = z_0(s),\ p = p_0(s),\ q = q_0(s),\ s_1 < s < s_2, \quad (2.17)$$

be a given *initial strip* such that the *strip condition*

$$z_0'(s) = p_0(s)x_0'(s) + q_0(s)y_0'(s) \quad (2.18)$$

is satisfied. Moreover, we assume that the initial strip satisfies the nonlinear equation, that is,

$$F(x_0(s), y_0(s), z_0(s), p_0(s), q_0(s)) = 0. \quad (2.19)$$

Initial value problem of Cauchy: *Find a C^2-solution $z = u(x,y)$ of $F(x,y,z,p,q) = 0$ such that the surface $\mathcal{S}$ defined by $z = u(x,y)$ containes the above initial strip.*

Similar to the quasilinear case we will show that the set of strips defined by the characteristic system which are sticked at the initial strip, see Figure 2.12, fit together and define the surface for which we are looking at.

Definition. A strip $(x(\tau), y(\tau), z(\tau), p(\tau), q(\tau))$, $\tau_1 < \tau < \tau_2$ is said to be *noncharacteristic* if

$$x'(\tau)F_q(x(\tau), y(\tau), z(\tau), p(\tau), q(\tau)) - y'(\tau)F_p(x(\tau), y(\tau), z(\tau), p(\tau), q(\tau)) \neq 0.$$

Theorem 2.2. *For a given noncharacteristic initial strip (2.17), $x_0,\ y_0,\ z_0 \in C^2$ and $p_0,\ q_0 \in C^1$ which satisfies the strip condition (2.18) and the differential equation (2.19) exists exactly one solution $z = u(x,y)$ of the Cauchy initial value problem in a neighbourhood of the initial curve $(x_0(s), y_0(s), z_0(s))$. That is, $z = u(x,y)$ is the solution of the differential equation (2.6) and $u(x_0(s), y_0(s)) = z_0(s)$, $u_x(x_0(s), y_0(s)) = p_0(s)$, $u_y(x_0(s), y_0(s)) = q_0(s)$.*

Proof. Consider the system (2.7)–(2.11) with the initial data

$$x(s,0) = x_0(s),\ y(s,0) = y_0(s),\ z(s,0) = z_0(s),\ p(s,0) = p_0(s),\ q(s,0) = q_0(s).$$

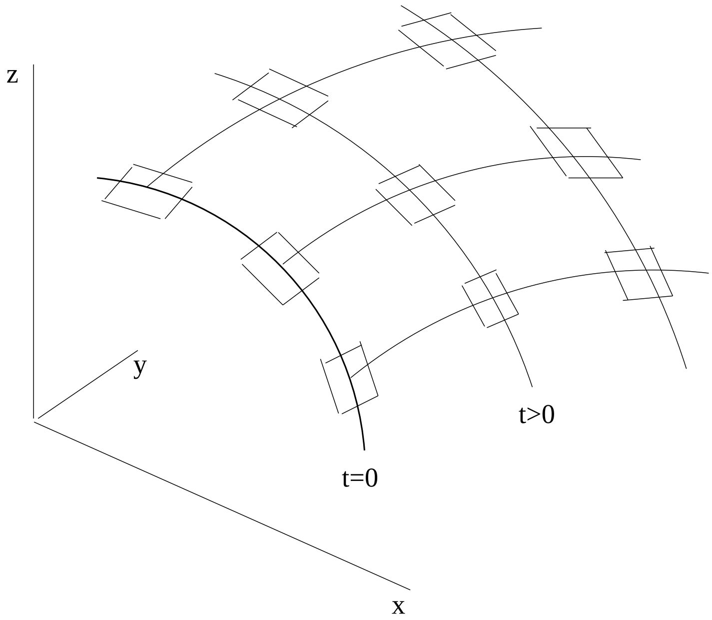

Figure 2.12: Construction of the solution

We will show that the surface defined by $x = x(s,t)$, $y(s,t)$ is the surface defined by $z = u(x,y)$, where u is the solution of the Cauchy initial value problem. It turns out that $u(x,y) = z(s(x,y),t(x,y))$, where $s = s(x,y)$, $t = t(x,y)$ is the inverse of $x = x(s,t)$, $y = y(s,t)$ in a neigbourhood of $t = 0$. This inverse exists since the initial strip is noncharacteristic by assumption:

$$\det \frac{\partial(x,y)}{\partial(s,t)}\Big|_{t=0} = x_0 F_q - y_0 F_q \neq 0.$$

Set

$$P(x,y) = p(s(x,y),t(x,y)), \quad Q(x,y) = q(s(x,y),t(x,y)).$$

From Proposition 2.3 and Proposition 2.4 it follows $F(x,y,u,P,Q) = 0$. We will show that $P(x,y) = u_x(x,y)$ and $Q(x,y) = u_y(x,y)$. To see this, we consider the function

$$h(s,t) = z_s - px_s - qy_s.$$

One has

$$h(s,0) = z_0'(s) - p_0(s)x_0'(s) - q_0(s)y_0'(s) = 0$$

since the initial strip satisfies the strip condition by assumption. In the following we will see that for fixed s the function h satisfies a linear homogeneous ordininary differential equation of first order. Consequently, $h(s,t)=0$ in a neighbourhood of $t=0$. That is, the strip condition is also satisfied along strips transversally to the characteristic strips, see Figure 2.18. That is, the set of "scales" fit together and define a surface like the scales of a fish.

From the definition of $h(s,t)$ and the characteristic equations it follows

$$\begin{aligned}
h_t(s,t) &= z_{st}-p_tx_s-q_ty_s-px_{st}-qy_{st}\\
&= \frac{\partial}{\partial s}(z_t-px_t-qy_t)+p_sx_t+q_sy_t-q_ty_s-p_tx_s\\
&= (px_s+qy_s)F_z+F_xx_s+F_yz_s+F_pp_s+F_qq_s.
\end{aligned}$$

Since $F(x(s,t),y(s,t),z(s,t),p(s,t),q(s,t))=0$, it follows after differentiation of this equation with respect to s the differential equation

$$h_t=-F_zh.$$

Hence $h(s,t)\equiv 0$, since $h(s,0)=0$.

Thus, we have

$$\begin{aligned}
z_s &= px_s+qy_s\\
z_t &= px_t+qy_t\\
z_s &= u_xx_s+u_yy_s\\
z_t &= u_xy_t+u_yy_t.
\end{aligned}$$

The first equation was shown above, the second is a characteristic equation and the last two follow from $z(s,t)=u(x(s,t),y(s,t))$. This system implies

$$\begin{aligned}
(P-u_x)x_s+(Q-u_y)y_s &= 0\\
(P-u_x)x_t+(Q-u_y)y_t &= 0.
\end{aligned}$$

It follows $P=u_x$ and $Q=u_y$.

The initial conditions

$$\begin{aligned}
u(x(s,0),y(s,0)) &= z_0(s)\\
u_x(x(s,0),y(s,0)) &= p_0(s)\\
u_y(x(s,0),y(s,0)) &= q_0(s)
\end{aligned}$$

are satisfied since

$$\begin{aligned}
u(x(s,t),y(s,t)) &= z(s(x,y),t(x,y)) = z(s,t) \\
u_x(x(s,t),y(s,t)) &= p(s(x,y),t(x,y)) = p(s,t) \\
u_y(x(s,t),y(s,t)) &= q(s(x,y),t(x,y)) = q(s,t).
\end{aligned}$$

The uniqueness follows as in the proof of Theorem 2.1. □

Example. A differential equation which occurs in the geometrical optic is

$$u_x^2 + u_y^2 = n(x,y),$$

where the positive function $n(x,y)$ is the index of refraction. The level sets defined by $u(x,y) = const.$ are called *wave fronts*. The characteristic curves $(x(t), y(t))$ are the rays of light. If n is a constant, then the rays of light are straight lines. In $\mathbb{R}^3$ the equation is

$$u_x^2 + u_y^2 + u_z^2 = n(x,y,z).$$

Thus we have to extend the previous theory from $\mathbb{R}^2$ to $\mathbb{R}^n$, $n \geq 3$.

2.4 Nonlinear equations in $\mathbb{R}^n$

Here we consider the nonlinear differential equation

$$F(x,z,p) = 0, \tag{2.20}$$

where

$$x = (x_1, \ldots, x_n),\ z = u(x):\ \Omega \subset \mathbb{R}^n \mapsto \mathbb{R},\ p = \nabla u.$$

The following system of $2n+1$ ordinary differential equations is said to be *characteristic system*.

$$\begin{aligned}
x'(t) &= \nabla_p F \\
z'(t) &= p \cdot \nabla_p F \\
p'(t) &= -\nabla_x F - F_z p.
\end{aligned}$$

Let

$$x_0(s) = (x_{01}, \ldots, x_{0n}),\ s = (s_1, \ldots, s_{n-1})$$

be a given regular (n-1)-dimensional C^2-hypersurface in $\mathbb{R}^n$, that is, we assume

$$\text{rank}\frac{\partial x_0(s)}{\partial s} = n-1.$$

Here is $s \in D$ a parameter from a $(n-1)$-dimensionl parameter domain.

For example, $x = x_0(s)$ defines in the three dimensional case a regular surface in $\mathbb{R}^3$.

Assume

$$z_0(s): \ D \mapsto \mathbb{R}, \ p_0(s) = (p_{01}(s), \ldots, p_{0n}(s))$$

are given sufficiently regular functions.

The $(2n+1)$-vector

$$(x_0(s), z_0(s), p_0(s))$$

is said to be *initial strip manifold* and the condition

$$\frac{\partial z_0}{\partial s_l} = \sum_{i=1}^{n-1} p_{0i}(s)\frac{\partial x_{0i}}{\partial s_l},$$

$l = 1, \ldots, n-1$, is called *strip condition.*

The initial strip manifold is said to be *noncharacteristic* if

$$\det\begin{pmatrix} F_{p_1} & F_{p_2} & \cdots & F_{p_n} \\ \frac{\partial x_{01}}{\partial s_1} & \frac{\partial x_{02}}{\partial s_1} & \cdots & \frac{\partial x_{0n}}{\partial s_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_{01}}{\partial s_{n-1}} & \frac{\partial x_{02}}{\partial s_{n-1}} & \cdots & \frac{\partial x_{0n}}{\partial s_{n-1}} \end{pmatrix} \neq 0,$$

where the argument of F_{p_j} is the initial strip manifold.

Initial value problem of Cauchy. *Seek a solution $z = u(x)$ of differential equation (2.20) such that the initial manifold is a subset of $\{(x, u(x), \nabla u(x)) : x \in \Omega\}$.*

As in the two dimensional case we have under additional regularity assumptions

Theorem 2.3. *Suppose the initial strip manifold is not characteristic and satisfies differential equation (2.20), that is, $F(x_0(s), z_0(s), p_0(s)) = 0$. Then there is a neighbourhood of the initial manifold $(x_0(s), z_0(s))$ such that there exists a unique solution of the Cauchy initial value problem.*

Sketch of proof. Let

$$x = x(s,t),\ z = z(s,t),\ p = p(s,t)$$

be the solution of the characteristic system and let

$$s = s(x),\ t = t(x)$$

be the inverse of $x = x(s,t)$ which exists in a neighbourhood of $t = 0$. Then, it turns out that

$$z = u(x) := z(s_1(x_1,\ldots,x_n),\ldots,s_{n-1}(x_1,\ldots,x_n),t(x_1,\ldots,x_n))$$

is the solution of the problem.

2.5 Hamilton-Jacobi theory

The nonlinear equation (2.20) of previous section in one more dimension is

$$F(x_1,\ldots,x_n,x_{n+1},z,p_1,\ldots,p_n,p_{n+1}) = 0.$$

The content of the Hamilton[1]-Jacobi[2] theory is the theory of the special case

$$F \equiv p_{n+1} + H(x_1,\ldots,x_n,x_{n+1},p_1,\ldots,p_n) = 0, \tag{2.21}$$

that is, the equation is linear in p_{n+1} and does not depend explicitely on z.

Remark. Formally, one can write equation (2.20)

$$F(x_1,\ldots,x_n,u,u_{x_1},\ldots,u_{x_n}) = 0$$

as an equation of type (2.21). Set $x_{n+1} = u$ and seek u implicitely from

$$\phi(x_1,\ldots,x_n,x_{n+1}) = const.,$$

where ϕ is a function which is defined by a differential equation.

[1]Hamilton, William Rowan, 1805–1865
[2]Jacobi, Carl Gustav, 1805–1851

Assume $\phi_{x_{n+1}} \neq 0$, then

$$\begin{aligned} 0 &= F(x_1,\ldots,x_n,u,u_{x_1},\ldots,u_{x_n}) \\ &= F(x_1,\ldots,x_n,x_{n+1},-\frac{\phi_{x_1}}{\phi_{x_{n+1}}},\ldots,-\frac{\phi_{x_n}}{\phi_{x_{n+1}}}) \\ &= : G(x_1,\ldots,x_{n+1},\phi_1,\ldots,\phi_{x_{n+1}}). \end{aligned}$$

Suppose that $G_{\phi_{x_{n+1}}} \neq 0$, then

$$\phi_{x_{n+1}} = H(x_1,\ldots,x_n,x_{n+1},\phi_{x_1},\ldots,\phi_{x_{n+1}}).$$

The associated characteristic equations to (2.21) are

$$\begin{aligned} x'_{n+1}(\tau) &= F_{p_{n+1}} = 1 \\ x'_k(\tau) &= F_{p_k} = H_{p_k}, \qquad k=1,\ldots,n \\ z'(\tau) &= \sum_{l=1}^{n+1} p_l F_{p_l} = \sum_{l=1}^{n} p_l H_{p_l} + p_{n+1} \\ &= \sum_{l=1}^{n} p_l H_{p_l} - H \\ p'_{n+1}(\tau) &= -F_{x_{n+1}} - F_z p_{n+1} \\ &= -F_{x_{n+1}} \\ p'_k(\tau) &= -F_{x_k} - F_z p_k \\ &= -F_{x_k}, \qquad k=1,\ldots,n. \end{aligned}$$

Set $t := x_{n+1}$, then we can write partial differential equation (2.21) as

$$u_t + H(x,t,\nabla_x u) = 0 \tag{2.22}$$

and $2n$ of the characteristic equations are

$$x'(t) = \nabla_p H(x,t,p) \tag{2.23}$$
$$p'(t) = -\nabla_x H(x,t,p). \tag{2.24}$$

Here is

$$x = (x_1,\ldots,x_n),\ p = (p_1,\ldots,p_n).$$

Let $x(t)$, $p(t)$ be a solution of (2.23) and (2.24), then it follows $p_{n+1}'(t)$ and $z'(t)$ from the characteristic equations

$$\begin{aligned} p_{n+1}'(t) &= -H_t \\ z'(t) &= p \cdot \nabla_p H - H. \end{aligned}$$

Definition. The function $H(x,t,p)$ is called *Hamilton function*, equation (2.21) *Hamilton-Jacobi equation* and the system (2.23), (2.24) *canonical system to H.*

There is an interesting interplay between the Hamilton-Jacobi equation and the canonical system. According to the previous theory we can construct a solution of the Hamilton-Jacobi equation by using solutions of the canonical system. On the other hand, one obtains from solutions of the Hamilton-Jacobi equation also solutions of the canonical system of ordinary differential equations.

Definition. A solution $\phi(a;x,t)$ of the Hamilton-Jacobi equation, where $a = (a_1,\ldots,a_n)$ is an n-tupel of real parameters, is called a *complete integral* of the Hamilton-Jacobi equation if

$$\det(\phi_{x_i a_l})_{i,l=1}^n \neq 0.$$

Remark. If u is a solution of the Hamilton-Jacobi equation, then also $u + const.$

Theorem 2.4 (Jacobi). *Assume*

$$u = \phi(a;x,t) + c,\ \ c = const.,\ \ \phi \in C^2 \ \textit{in its arguments},$$

is a complete integral. Then one obtaines by solving of

$$b_i = \phi_{a_i}(a;x,t),\ \ b_i\ i = 1,\ldots,n,\ \ \textit{are given real constants},$$

with respect to $x_l = x_l(a,b,t)$ *and then by setting*

$$p_k = \phi_{x_k}(a;x(a,b;t),t)$$

a 2n-parameter family of solutions of the canonical system.

Proof. Let

$$x_l(a,b;t),\ l=1,\ldots,n$$

be the solution of the above system. The solution exists since ϕ is a complete integral by assumption. Set

$$p_k(a,b;t)=\phi_{x_k}(a;x(a,b;t),t),\ k=1,\ldots,n.$$

We will show that x and p solves the canonical system. Differentiating $\phi_{a_i}=b_i$ with respect to t and the Hamilton-Jacobi equation $\phi_t+H(x,t,\nabla_x\phi)=0$ with respect to a_i, we obtain for $i=1,\ldots,n$

$$\begin{aligned} \phi_{ta_i}+\sum_{k=1}^n \phi_{x_k a_i}\frac{\partial x_k}{\partial t} &= 0 \\ \phi_{ta_i}+\sum_{k=1}^n \phi_{x_k a_i} H p_k &= 0. \end{aligned}$$

Since ϕ is a complete integral it follows for $k=1,\ldots,n$

$$\frac{\partial x_k}{\partial t}=H_{p_k}.$$

Along a trajectory, that is, where $a,\ b$ are fixed, it is $\frac{\partial x_k}{\partial t}=x_k'(t)$. Thus

$$x_k'(t)=H_{p_k}.$$

Now we differentiate $p_i(a,b;t)$ with respect to t and $\phi_t+H(x,t,\nabla_x\phi)=0$ with respect to x_i, and obtain

$$\begin{aligned} p_i'(t) &= \phi_{x_i t}+\sum_{k=1}^n \phi_{x_i x_k} x_k'(t) \\ 0 &= \phi_{x_i t}+\sum_{k=1}^n \phi_{x_i x_k} H_{p_k}+H_{x_i} \\ 0 &= \phi_{x_i t}+\sum_{k=1}^n \phi_{x_i x_k} x_k'(t)+H_{x_i} \end{aligned}$$

It follows finally $p_i'(t)=-H_{x_i}$. □

Example: **Kepler problem**

The motion of a mass point in a central field takes place in a plane, say the (x, y)-plane, see Figure 2.13, and satisfies the system of ordinary differential equations of second order

$$x''(t) = U_x,\ y''(t) = U_y,$$

where

$$U(x, y) = \frac{k^2}{\sqrt{x^2 + y^2}}.$$

Here we assume that k^2 is a positive constant and that the mass point is attracted of the origin. In the case that it is pushed one has to replace U by $-U$. See Landau and Lifschitz [9], Vol 1, for example, for the related physics.

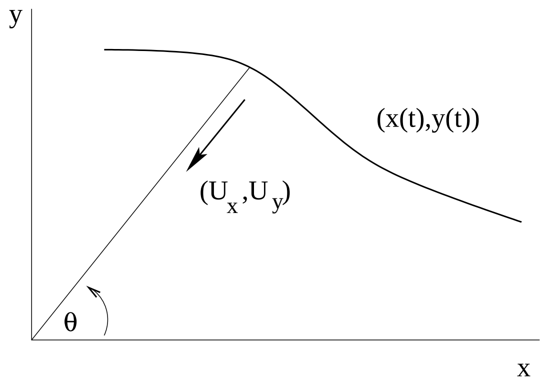

Figure 2.13: Motion in a central field

Set

$$p = x',\ q = y'$$

and

$$H = \frac{1}{2}(p^2 + q^2) - U(x, y),$$

then

$$\begin{aligned} x'(t) &= H_p,\ y'(t) = H_q \\ p'(t) &= -H_x,\ q'(t) = -H_y. \end{aligned}$$

The associated Hamilton-Jacobi equation is

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) = \frac{k^2}{\sqrt{x^2+y^2}}.$$

which is in polar coordines (r, θ)

$$\phi_t + \frac{1}{2}(\phi_r^2 + \frac{1}{r^2}\phi_\theta^2) = \frac{k^2}{r}. \tag{2.25}$$

Now, we will seek a complete integral of (2.25) by making the ansatz

$$\phi_t = -\alpha = const. \quad \phi_\theta = -\beta = const. \tag{2.26}$$

and obtain from (2.25) that

$$\phi = \pm \int_{r_0}^{r} \sqrt{2\alpha + \frac{2k^2}{\rho} - \frac{\beta^2}{\rho^2}}\, d\rho + c(t, \theta).$$

From ansatz (2.26) it follows

$$c(t, \theta) = -\alpha t - \beta\theta.$$

Therefore we have a two parameter family of solutions

$$\phi = \phi(\alpha, \beta; \theta, r, t)$$

of the Hamilton-Jacobi equation. This solution is a complete integral (exercise). According to the theorem of Jacobi set

$$\phi_\alpha = -t_0, \quad \phi_\beta = -\theta_0.$$

Then

$$t - t_0 = -\int_{r_0}^{r} \frac{d\rho}{\sqrt{2\alpha + \frac{2k^2}{\rho} - \frac{\beta^2}{\rho^2}}}.$$

The inverse function $r = r(t)$, $r(0) = r_0$, is the r-coordinate depending on time t, and

$$\theta - \theta_0 = \beta \int_{r_0}^{r} \frac{d\rho}{\rho^2\sqrt{2\alpha + \frac{2k^2}{\rho} - \frac{\beta^2}{\rho^2}}}.$$

Substitution $\tau = \rho^{-1}$ yields

$$\begin{aligned}
\theta - \theta_0 &= -\beta \int_{1/r_0}^{1/r} \frac{d\tau}{\sqrt{2\alpha + 2k^2\tau - \beta^2\tau^2}} \\
&= -\arcsin\left(\frac{\frac{\beta^2}{k^2}\frac{1}{r} - 1}{\sqrt{1+\frac{2\alpha\beta^2}{k^4}}}\right) + \arcsin\left(\frac{\frac{\beta^2}{k^2}\frac{1}{r_0} - 1}{\sqrt{1+\frac{2\alpha\beta^2}{k^4}}}\right).
\end{aligned}$$

Set

$$\theta_1 = \theta_0 + \arcsin\left(\frac{\frac{\beta^2}{k^2}\frac{1}{r_0} - 1}{\sqrt{1+\frac{2\alpha\beta^2}{k^4}}}\right)$$

and

$$p = \frac{\beta^2}{k^2}, \quad \epsilon^2 = \sqrt{1 + \frac{2\alpha\beta^2}{k^4}},$$

then

$$\theta - \theta_1 = -\arcsin\left(\frac{\frac{p}{r} - 1}{\epsilon^2}\right).$$

It follows

$$r = r(\theta) = \frac{p}{1 - \epsilon^2 \sin(\theta - \theta_1)},$$

which is the polar equation of conic sections. It defines an ellipse if $0 \le \epsilon < 1$, a parabola if $\epsilon = 1$ and a hyperbola if $\epsilon > 1$, see Figure 2.14 for the case of an ellipse, where the origin of the coordinate system is one of the focal points of the ellipse.

For another applicaton of the Jacobi theorem see Courant and Hilbert [4], Vol. 2, pp. 94, where geodedics on an ellipsoid are studied.

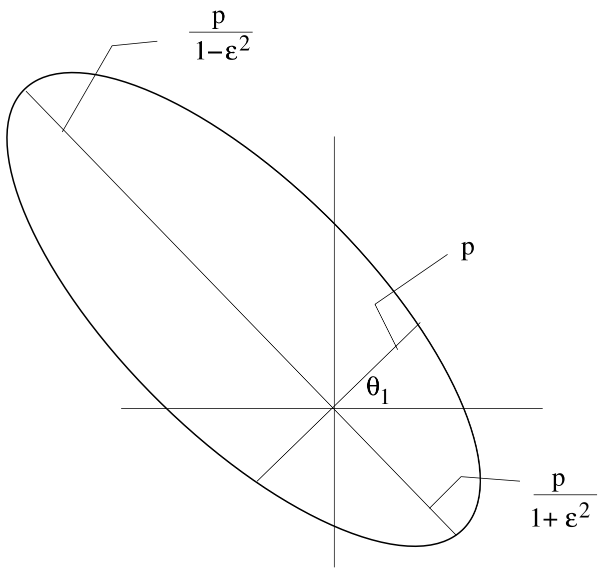

Figure 2.14: The case of an ellipse

2.6 Exercises

1. Suppose $u: \mathbb{R}^2 \mapsto \mathbb{R}$ is a solution of

$$a(x,y)u_x + b(x,y)u_y = 0.$$

   Show that for arbitrary $H \in C^1$ also $H(u)$ is a solution.

2. Find a solution $u \not\equiv const.$ of

$$u_x + u_y = 0$$

   such that

$$\text{graph}(u) := \{(x,y,z) \in \mathbb{R}^3 : \ z = u(x,y), \ (x,y) \in \mathbb{R}^2\}$$

   contains the straight line $(0,0,1) + s(1,1,0), \ s \in \mathbb{R}$.

3. Let $\phi(x,y)$ be a solution of

$$a_1(x,y)u_x + a_2(x,y)u_y = 0 \ .$$

   Prove that level curves $S_C := \{(x,y) : \ \phi(x,y) = C = const.\}$ are characteristic curves, provided that $\nabla\phi \neq 0$ and $(a_1,a_2) \neq (0,0)$.

4. Prove Proposition 2.2.

5. Find two different solutions of the initial value problem

$$u_x + u_y = 1,$$

where the initial data are $x_0(s) = s,\ y_0(s) = s,\ z_0(s) = s$.

Hint. (x_0, y_0) is a characteristic curve.

6. Solve the initial value problem

$$xu_x + yu_y = u$$

with initial data $x_0(s) = s,\ y_0(s) = 1,\ z_0(s)$, where z_0 is given.

7. Solve the initial value problem

$$-xu_x + yu_y = xu^2,$$

$x_0(s) = s,\ y_0(s) = 1,\ z_0(s) = \mathrm{e}^{-s}$.

8. Solve the initial value problem

$$uu_x + u_y = 1,$$

$x_0(s) = s,\ y_0(s) = s,\ z_0(s) = s/2$ if $0 < s < 1$.

9. Solve the initial value problem $u_x^2 + u_y^2 = 1 + x$ with given initial data $x_0(s) = 0,\ y_0(s) = s,\ u_0(s) = 1,\ p_0(s) = 1,\ q_0(s) = 0,\ -\infty < s < \infty$.

10. Find the solution $\Phi(x, y)$ of

$$(x - y)u_x + 2yu_y = 3x\ ,$$

such that the surface defined by $z = \Phi(x, y)$ contains the curve

$$C: \quad x_0(s) = s,\ y_0(s) = 1,\ z_0(s) = 0,\ s \in \mathbb{R}.$$

11. Solve the following initial problem of chemical kinetics.

$$u_x + u_y = \left(k_0\mathrm{e}^{-k_1 x} + k_2\right)(1 - u)^2,\ x > 0,\ y > 0$$

with the initial data $u(x, 0) = 0,\ u(0, y) = u_0(y)$, where u_0, $0 < u_0 < 1$, is given.

12. Solve

$$
\begin{aligned}
u_{x_1} + u_{x_2} &= 0 \\
u(x_1, 0) &= g(x_1)
\end{aligned}
$$

in $\Omega_1 = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 > x_2\}$ and in $\Omega_2 = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 < x_2\}$, where

$$
g(x_1) = \begin{cases} u_l & : \quad x_1 < 0 \\ u_r & : \quad x_1 > 0 \end{cases}
$$

with constants $u_l \neq u_r$.

Remark. Such a problem with discontinuous initial data is called Riemann's problem.

13. Determine the opening angle of the Monge cone, that is, the angle between the axis and the apothem (german: Mantellinie) of the cone, for equation

$$
u_x^2 + u_y^2 = f(x, y, u),
$$

where $f > 0$.

14. Prove: $F(x, y, u, p, q)$ is an integral, that is, $F(x, y, u, p, q)$ is constant along each characteristic curve $(x(t), y(t), z(t), p(t), q(t))$.

15. Solve the initial value problem

$$
u_x^2 + u_y^2 = 1,
$$

where $x_0(\theta) = a\cos\theta,\ y_0(\theta) = a\sin\theta,\ z_0(\theta) = 1,\ p_0(\theta) = \cos\theta,\ q_0(\theta) = \sin\theta$ if $0 \le \theta < 2\pi$, $a = const. > 0$.

16. Show that the integral $\phi(\alpha, \beta; \theta, r, t)$, see the Kepler problem, is a complete integral.

17. a) Show that $S = \sqrt{\alpha}\ x + \sqrt{1-\alpha}\ y + \beta$, $\alpha,\ \beta \in \mathbb{R},\ 0 < \alpha < 1$, is a complete integral of $S_x - \sqrt{1 - S_y^2} = 0$.
b) Find the envelope of this family of solutions.

18. Determine the length of the half axis of the ellipse

$$
r = \frac{p}{1 - \varepsilon^2 \sin(\theta - \theta_0)}, \quad 0 \le \varepsilon < 1.
$$

19. Find the Hamilton function $H(x,p)$ of the Hamilton-Jacobi-Bellman differential equation if $h = 0$ and $f = Ax + B\alpha$, where A, B are constant and real matrices, $A:\ \mathbb{R}^m \mapsto \mathbb{R}^n$, B is an orthogonal real $n \times n$-Matrix and $p \in \mathbb{R}^n$ is given. The set of admissible controls is given by
$$U = \{\alpha \in \mathbb{R}^n :\ \sum_{i=1}^{n} \alpha_i^2 \leq 1\}\ .$$

    Remark. The Hamilton-Jacobi-Bellman equation is formally the Hamilton-Jacobi equation $u_t + H(x, \nabla u) = 0$, where the Hamilton function is defined by
$$H(x,p) := \min_{\alpha \in U} \left(f(x,\alpha) \cdot p + h(x,\alpha) \right),$$
    $f(x,\alpha)$ and $h(x,\alpha)$ are given. See for example, Evans[5], Chapter 10.

Chapter 3

Classification

Different types of problems in physics, for example, correspond different types of partial differential equations. The methods how to solve these equations differ from type to type.

The classification of differential equations follows from one single question: Can we calculate formally the solution if sufficiently many initial data are given? Consider the initial problem for an ordinary differential equation $y'(x) = f(x, y(x))$, $y(x_0) = y_0$. Then one can determine formally the solution, provided the function $f(x, y)$ is sufficiently regular. The solution of the initial value problem is formally given by a power series. This formal solution is a solution of the problem if $f(x, y)$ is real analytic according to a theorem of Cauchy. In the case of partial differential equations the related theorem is the Theorem of Cauchy-Kowalevski. Even in the case of ordinary differential equations the situation is more complicated if y' is implicitely defined, that is, the differential equation is $F(x, y(x), y'(x)) = 0$ for a given function F.

3.1 Linear equations of second order

The general nonlinear partial differential equation of second order is

$$F(x, u, Du, D^2u) = 0,$$

where $x \in \mathbb{R}^n$, $u : \ \Omega \subset \mathbb{R}^n \mapsto \mathbb{R}$, $Du \equiv \nabla u$ and D^2u stands for all second derivatives. The function F is given and sufficiently regular with respect to its $2n + 1 + n^2$ arguments.

In this section we consider the case

$$\sum_{i,k=1}^{n} a_{ik}(x)u_{x_ix_k} + f(x,u,\nabla u) = 0. \tag{3.1}$$

The equation is *linear* if

$$f = \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u + d(x).$$

Concerning the classification the *main part*

$$\sum_{i,k=1}^{n} a_{ik}(x)u_{x_ix_k}$$

plays the essential role. Suppose $u \in C^2$, then we can assume, without restriction of generality, that $a_{ik} = a_{ki}$, since

$$\sum_{i,k=1}^{n} a_{ik}(x)u_{x_ix_k} = \sum_{i,k=1}^{n} a^\star_{ik}(x)u_{x_ix_k},$$

where

$$a^\star_{ik} = \frac{1}{2}(a_{ik} + a_{ki}).$$

Consider a hypersurface $\mathcal{S}$ in $\mathbb{R}^n$ defined implicitely by $\chi(x) = 0$, $\nabla\chi \neq 0$, see Figure 3.1

Assume u and ∇u are given on $\mathcal{S}$.

Problem: *Can we calculate all other derivatives of u on $\mathcal{S}$ by using differential equation (3.1) and the given data?*

We will find an answer if we map $\mathcal{S}$ onto a hyperplane $\mathcal{S}_0$ by a mapping

$$\begin{aligned}\lambda_n &= \chi(x_1,\ldots,x_n)\\ \lambda_i &= \lambda_i(x_1,\ldots,x_n),\ i = 1,\ldots,n-1,\end{aligned}$$

for functions λ_i such that

$$\det\frac{\partial(\lambda_1,\ldots,\lambda_n)}{\partial(x_1,\ldots,x_n)} \neq 0$$

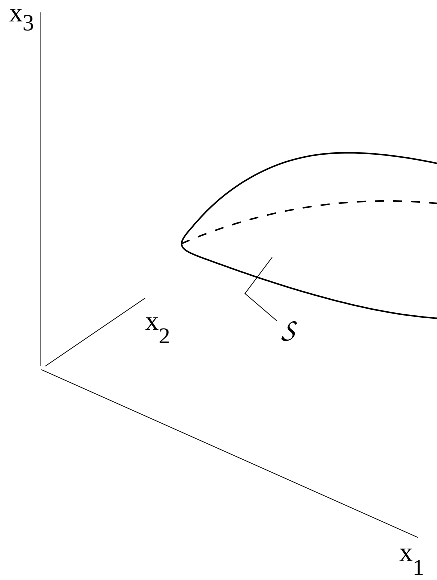

Figure 3.1: Initial manifold $\mathcal{S}$

in $\Omega \subset \mathbb{R}^n$. It is assumed that χ and λ_i are sufficiently regular. Such a mapping $\lambda = \lambda(x)$ exists, see an exercise.

The above transformation maps $\mathcal{S}$ onto a subset of the hyperplane defined by $\lambda_n = 0$, see Figure 3.2

We will write the differential equation in these new coordinates. Here we use Einstein's convention, that is, we add terms with repeating indices. Since

$$u(x) = u(x(\lambda)) =: v(\lambda) = v(\lambda(x)),$$

where $x = (x_1, \ldots, x_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$, we get

$$\begin{aligned} u_{x_j} &= v_{\lambda_i}\frac{\partial \lambda_i}{\partial x_j}, \\ u_{x_j x_k} &= v_{\lambda_i \lambda_l}\frac{\partial \lambda_i}{\partial x_j}\frac{\partial \lambda_l}{\partial x_k} + v_{\lambda_i}\frac{\partial^2 \lambda_i}{\partial x_j \partial x_k}. \end{aligned} \tag{3.2}$$

Thus, differential equation (3.1) in the new coordinates is given by

$$a_{jk}(x)\frac{\partial \lambda_i}{\partial x_j}\frac{\partial \lambda_l}{\partial x_k}v_{\lambda_i \lambda_l} + \text{terms known on } \mathcal{S}_0 = 0.$$

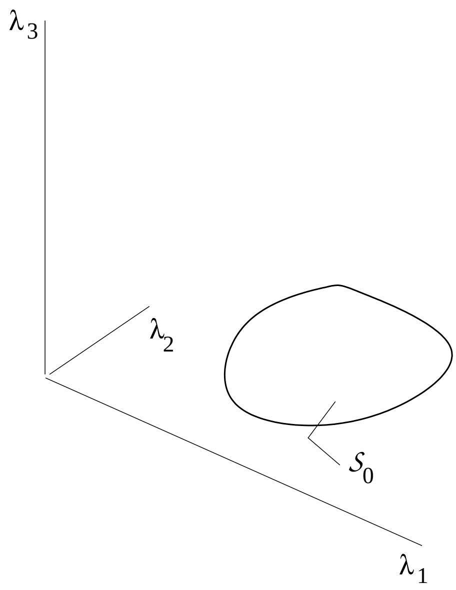

Figure 3.2: Transformed flat manifold $\mathcal{S}_0$

Since $v_{\lambda_k}(\lambda_1,\ldots,\lambda_{n-1},0)$, $k=1,\ldots,n$ are known, see (3.2), it follows that $v_{\lambda_k\lambda_l}$, $l=1,\ldots,n{-}1$ are known on $\mathcal{S}_0$. That is, we know all second derivatives $v_{\lambda_i\lambda_j}$ on $\mathcal{S}_0$ with the only exception of $v_{\lambda_n\lambda_n}$.

We recall that, provided v is sufficiently regular,

$$v_{\lambda_k\lambda_l}(\lambda_1,\ldots,\lambda_{n-1},0)$$

is the limit of

$$\frac{v_{\lambda_k}(\lambda_1,\ldots,\lambda_l+h,\lambda_{l+1},\ldots,\lambda_{n-1},0)-v_{\lambda_k}(\lambda_1,\ldots,\lambda_l,\lambda_{l+1},\ldots,\lambda_{n-1},0)}{h}$$

as $h\to 0$.

Thus, the differential equation is now

$$a_{jk}(x)\frac{\partial\lambda_n}{\partial x_j}\frac{\partial\lambda_n}{\partial x_k}v_{\lambda_n\lambda_n}=\text{terms known on }\mathcal{S}_0.$$

It follows that we can calculate $v_{\lambda_n\lambda_n}$ if

$$\sum_{i,j=1}^{n}a_{ij}(x)\chi_{x_i}\chi_{x_j}\neq 0 \tag{3.3}$$

on $\mathcal{S}$. This is a condition for the given equation and for the given surface $\mathcal{S}$.

Definition. Differential equation

$$\sum_{i,j=1}^{n} a_{ij}(x)\chi_{x_i}\chi_{x_j} = 0$$

is called *characteristic differential equation* associated to the given differential equation (3.1).

If χ, $\nabla\chi \neq 0$, is a solution of the characteristic differential equation, then the surface defined by $\chi = 0$ is called *characteristic surface*.

Remark. The condition (3.3) is satisfied for each χ with $\nabla\chi \neq 0$ if the quadratic matrix $(a_{ij}(x))$ is positive or negative definite for each $x \in \Omega$, which is aquivalent to the property that all eigenvalues are different from zero and from the same sign. This follows since there is a $\lambda(x) > 0$ such that, in the case that (a_{ij}) is poitive definite,

$$a_{ij}(x)\zeta_i\zeta_j \geq \lambda(x)|\zeta|^2$$

for all $\zeta \in \mathbb{R}^n$. Here and in the following we assume that the matrix (a_{ij}) is real and symmetric.

The characterization of differential equation (3.1) follows from the signs of the eigenvalues of $(a_{ij}(x))$.

Definition. Differential equation (3.1) is said to be of *type* (α, β, γ) *at* $x \in \Omega$ if α eigenvalues of $(a_{ij})(x)$ are positive, β eigenvalues are negative and γ eigenvalues are zero ($\alpha + \beta + \gamma = n$).
In particular, equation is called
elliptic if it is of type $(n, 0, 0)$ or of type $(0, n, 0)$, that is all eigenvalues are different from zero and have the same sign.
parabolic if it is of type $(n-1, 0, 1)$ or of type $(0, n-1, 1)$, that is one eigenvalue is zero and all the others are different from zero and have the same sign.
hyperbolic if it is of type $(n-1, 1, 0)$ or of type $(1, n-1, 0)$, that is, all eigenvalues are different from zero and one eigenvalue has another sign than all the others.

Remarks:

1. According to this definition there are other types aside from elliptic, parabolic or hyperbolic equations.

2. The classification depends in general on $x \in \Omega$. An example is the Tricomi equation, which appears in the theory of transsonic flows,

$$yu_{xx} + u_{yy} = 0.$$

This equation is elliptic if $y > 0$, parabolic if $y = 0$ and hyperbolic for $y < 0$.

Examples:

1. The *Laplace equation* in $\mathbb{R}^3$ is $\triangle u = 0$, where

$$\triangle u := u_{xx} + u_{yy} + u_{zz}.$$

This equation is elliptic. That is, for each mannifold $\mathcal{S}$ given by $\{(x, y, z) : \chi(x, y, z) = 0\}$, where χ is an arbitrary sufficiently regular function such that $\nabla\chi \neq 0$, all derivatives of u are known on $\mathcal{S}$, provided u and ∇u are known on $\mathcal{S}$.

2. The *wave equation* $u_{tt} = u_{xx} + u_{yy} + u_{zz}$, where $u = u(t, x, y, z)$, is hyperbolic. Such type describes oscillations of mechanical structures, for example.

3. The *heat equation* $u_t = u_{xx} + u_{yy} + u_{zz}$, where $u = u(t, x, y, z)$, is parabolic. It describes, for example, the propagation of heat in a domain.

4. Consider the case that the (real) coefficients a_{ij} in equation (3.1) are *constant*. We recall that the matrix $A = (a_{ij})$ is symmetric, that is $A^T = A$. In this case, the transformation to principle axis leads to a normal form from which the classification of the equation is obviously. Let U be the associated orthogonal matrix, that is,

$$U^T AU = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Here is $U = (z_1, \ldots, z_n)$, where z_l, $l = 1, \ldots, n$, is an orthonormal system of eigenvectors to the eigenvalues λ_l.

Set $y = U^T x$ and $v(y) = u(Uy)$, then

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} = \sum_{i=1}^{n} \lambda_i v_{y_i y_j}. \tag{3.4}$$

3.1.1 Normal form in two variables

Consider differential equation

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + \text{terms of lower order} = 0 \tag{3.5}$$

in $\Omega \subset \mathbb{R}^2$. The associated characteristic differential equation is

$$a\chi_x^2 + 2b\chi_x\chi_y + c\chi_y^2 = 0. \tag{3.6}$$

We show that an appropriate coordinate transformation will simplify equation (3.5), sometimes in such a way that we can solve it explicitely.

Assume there is a solution $z = \phi(x,y)$ of (3.6). Consider the level sets $\{(x,y):\ \phi(x,y) = const.\}$ and assume that $\phi_y \neq 0$ at a point (x_0, y_0) of the level set. Consequently, there is a function $y(x)$ defined in a neighbourhood of x_0 such that $\phi(x, y(x)) = const.$. It follows

$$y'(x) = -\frac{\phi_x}{\phi_y},$$

which implies, see the characteristic equation (3.6),

$$ay'^2 - 2by' + c = 0. \tag{3.7}$$

That is, provided that $a \neq 0$, we can calculate $\mu := y'$ from the (known) coefficients a, b and c:

$$\mu_{1,2} = \frac{1}{a}\left(b \pm \sqrt{b^2 - ac}\right). \tag{3.8}$$

These solutions are real if and only of $ac - b^2 \leq 0$.

Equation (3.5) is hyperbolic if $ac - b^2 < 0$, parabolic if $ac - b^2 = 0$ and elliptic if $ac - b^2 > 0$. This follows from an easy discussion of the eigenvalues of the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

see an exercise.

Normal form of a hyperbolic equation

Let ϕ and ψ are solutions of the characteristic equation (3.6) such that

$$\begin{aligned} y_1' \equiv \mu_1 &= -\frac{\phi_x}{\phi_y} \\ y_2' \equiv \mu_2 &= -\frac{\psi_x}{\psi_y}, \end{aligned}$$

where μ_1 and μ_2 are given by (3.8). Thus ϕ and ψ are solutions of the linear homogeneous equations of first order

$$\phi_x + \mu_1(x,y)\phi_y = 0 \tag{3.9}$$
$$\psi_x + \mu_2(x,y)\psi_y = 0. \tag{3.10}$$

Consider solutions $\phi(x,y)$, $\psi(x,y)$ such that $\nabla\phi \neq 0$ and $\nabla\psi \neq 0$, see an exercise for the existence of such solutions.

Consider two families of level sets defined by $\phi(x,y) = \alpha$ and $\psi(x,y) = \beta$, see Figure 3.3.

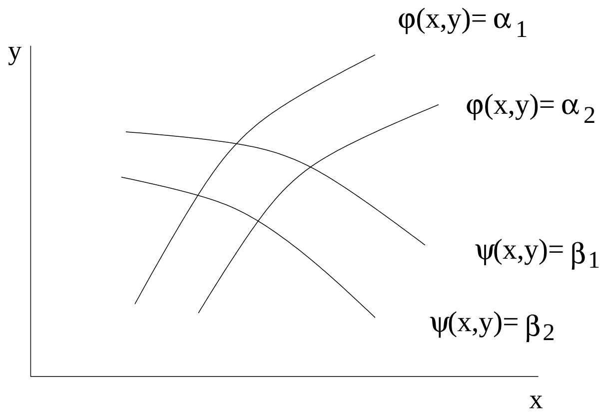

Figure 3.3: Level sets

These level sets are characteristic curves of the partial differential equations (3.9) and (3.10), respectively, see an exercise of the previous chapter.

Lemma. (i) *Curves from different families can not touch each other.*

(ii) $\phi_x\psi_y - \phi_y\psi_x \neq 0$.

Proof. (i):

$$y_2' - y_1' \equiv \mu_2 - \mu_1 = -\frac{2}{a}\sqrt{b^2 - ac} \neq 0.$$

(ii):

$$\mu_2 - \mu_1 = \frac{\phi_x}{\phi_y} - \frac{\psi_x}{\psi_y}.$$

□

Proposition. *The mapping $\xi = \phi(x,y)$, $\eta = \psi(x,y)$ transforms equation (3.5) into*

$$v_{\xi\eta} = \textit{lower order terms}, \tag{3.11}$$

where $v(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$.

Proof. The proof follows from a straightforward calculation.

$$\begin{aligned}
u_x &= v_\xi \phi_x + v_\eta \psi_x \\
u_y &= v_\xi \phi_y + v_\eta \psi_y \\
u_{xx} &= v_{\xi\xi}\phi_x^2 + 2v_{\xi\eta}\phi_x\psi_x + v_{\eta\eta}\psi_x^2 + \text{lower order terms} \\
u_{xy} &= v_{\xi\xi}\phi_x\phi_y + v_{\xi\eta}(\phi_x\psi_y + \phi_y\psi_x) + v_{\eta\eta}\psi_x\psi_y + \text{lower order terms} \\
u_{yy} &= v_{\xi\xi}\phi_y^2 + 2v_{\xi\eta}\phi_y\psi_y + v_{\eta\eta}\psi_y^2 + \text{lower order terms.}
\end{aligned}$$

It follows

$$au_{xx} + 2bu_{xy} + cu_{yy} = \alpha v_{\xi\xi} + 2\beta v_{\xi\eta} + \gamma v_{\eta\eta} + l.o.t.,$$

where

$$\begin{aligned}
\alpha &= a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 \\
\beta &= a\phi_x\psi_x + b(\phi_x\psi_y + \phi_y\psi_x) + c\phi_y\psi_y \\
\gamma &= a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2.
\end{aligned}$$

The coefficients α and γ are zero since ϕ and ψ are solutions of the characteristic equation. Since

$$\alpha\gamma - \beta^2 = (ac - b^2)(\phi_x\psi_y - \phi_y\psi_x)^2,$$

it follows from the above lemma that the coefficient β is different from zero.

□

Example: Consider differential equation

$$u_{xx} - u_{yy} = 0.$$

The associated characteristic differential equation is

$$\chi_x^2 - \chi_y^2 = 0.$$

Since $\mu_1 = -1$ and $\mu_2 = 1$, the fuctions ϕ and ψ satisfy differential equations

$$\begin{aligned} \phi_x + \phi_y &= 0 \\ \psi_x - \psi_y &= 0. \end{aligned}$$

Solutions with $\nabla\phi \neq 0$ and $\nabla\psi \neq 0$ are

$$\phi = x - y, \quad \psi = x + y.$$

Thus, the mapping

$$\xi = x - y, \quad \eta = x + y$$

leads to the simple equation

$$v_{\xi\eta}(\xi, \eta) = 0.$$

Assume that $v \in C^2$ is a solution, then $v_\xi = f_1(\xi)$ for an arbitrary C^1 function $f_1(\xi)$. It follows

$$v(\xi, \eta) = \int_0^\xi f_1(\alpha)\ d\alpha + g(\eta),$$

where g is an arbitrary C^2 function. That is, each C^2 solution of the differential equation can be written as

$$(\star) \qquad v(\xi, \eta) = f(\xi) + g(\eta),$$

where $f,\ g \in C^2$. On the other hand, for arbitrary C^2 functions the function $(\star)$ is a solution of the differential equation $v_{\xi\eta} = 0$. Consequently, each C^2 solution of the original equation $u_{xx} - u_{yy} = 0$ is given by

$$u(x, y) = f(x - y) + g(x + y),$$

where $f,\ g \in C^2$.

3.2 Quasilinear equations of second order

Here we consider equation

$$\sum_{i,j=1}^{n} a_{ij}(x,u,\nabla u)u_{x_i x_j} + b(x,u,\nabla u) = 0 \tag{3.12}$$

in a domain $\Omega \subset \mathbb{R}^n$, where $u:\ \Omega \mapsto \mathbb{R}$. We assume that $a_{ij} = a_{ji}$.

As in the previous section we derive the characteristic equation

$$\sum_{i,j=1}^{n} a_{ij}(x,u,\nabla u)\chi_{x_i}\chi_{x_j} = 0.$$

In contrast to linear equations, solutions of the characteristic equation depends on the solution considered.

3.2.1 Quasilinear elliptic equations

There is a large class of quasilinear equations such that the associated characteristic equation has no solution χ, $\nabla\chi \neq 0$.

Set

$$U = \{(x,z,p) :\ x \in \Omega,\ z \in \mathbb{R},\ p \in \mathbb{R}^n\}.$$

Definition. The quasilinear equation (3.12) is called *elliptic* if the matrix $(a_{ij}(x,z,p))$ is positive definite for each $(x,z,p) \in U$.

Assume equation (3.12) is elliptic and let $\lambda(x,z,p)$ be the minimum and $\Lambda(x,z,p)$ the maximum of the eigenvalues of (a_{ij}), then

$$0 < \lambda(x,z,p)|\zeta|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,z,p)\zeta_i\zeta_j \leq \Lambda(x,z,p)|\zeta|^2$$

for all $\zeta \in \mathbb{R}^n$.

Definition. Equation (3.12) is *uniformly elliptic* if Λ/λ is uniformly bounded in U.

An important class of elliptic equations which are not uniformly elliptic (non-uniformly elliptic) is

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}}\right) + \text{lower order terms} = 0. \tag{3.13}$$

That is, the main part is the minimal surface operator (left hand side of the minimal surface equation). The coefficients a_{ij} are

$$a_{ij}(x,z,p) = \left(1+|p|^2\right)^{-1/2}\left(\delta_{ij} - \frac{p_i p_j}{1+|p|^2}\right),$$

δ_{ij} denotes the Kronecker delta symbol. It follows that

$$\lambda = \frac{1}{(1+|p|^2)^{3/2}}, \quad \Lambda = \frac{1}{(1+|p|^2)^{1/2}}.$$

Thus, equation (3.13) is not uniformly elliptic.

The behaviour of solutions of uniformly elliptic equations is similar to linear elliptic equations in contrast to the behaviour of solutions of non-uniformly elliptic equations. Typical examples for non-uniformly elliptic equations are the minimal surface equation and the capillary equation.

3.3 Systems of first order

Consider the quasilinear system

$$\sum_{k=1}^{n} A^k(x,u)u_{u_k} + b(x,u) = 0, \tag{3.14}$$

where A^k are $m \times m$-matrices, sufficiently regular with respects to their arguments, and

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \quad u_{x_k} = \begin{pmatrix} u_{1,x_k} \\ \vdots \\ u_{m,x_k} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

We ask the same question as above: Can we calculate all derivatives of u in a neighbourhood of a given hypersurface $\mathcal{S}$ in $\mathbb{R}^n$ defined by $\chi(x) = 0$, $\nabla\chi \neq 0$, provided $u(x)$ is given on $\mathcal{S}$?

For an answer we map $\mathcal{S}$ on a flat surface $\mathcal{S}_0$ by using the mapping $\lambda = \lambda(x)$ of Section 3.1 and write equation (3.14) in new coordinates. Set $v(\lambda) = u(x(\lambda))$, then

$$\sum_{k=1}^{n} A^k(x,u)\chi_{x_k} v_{\lambda_n} = \text{terms known on } \mathcal{S}_0.$$

That is, we can solve this system with respect to v_{λ_n}, provided that

$$\det\left(\sum_{k=1}^{n} A^k(x,u)\chi_{x_k}\right) \neq 0$$

on $\mathcal{S}$.

Definition. Equation

$$\det\left(\sum_{k=1}^{n} A^k(x,u)\chi_{x_k}\right) = 0$$

is called *characteristic equation* associated to equation (3.14) and a surface $\mathcal{S}$: $\xi(x) = 0$, defined by a solution ξ, $\nabla\chi \neq 0$, of this characteristic equation is said to be *characteristic surface*.

Set

$$C(x,u,\zeta) = \det\left(\sum_{k=1}^{n} A^k(x,u)\zeta_k\right)$$

for $\zeta \in \mathbb{R}^n$ and define

Definition. (i) The system (3.14) is *hyperbolic* at $(x, u(x))$, if there is a regular linear mapping $\zeta = Q\eta$, where $\eta = (\eta_1, \ldots, \eta_{n-1}, \kappa)$, such that there exists m *real* roots $\kappa_k = \kappa_k(x, u(x), \eta_1, \ldots, \eta_{n-1})$, $k = 1, \ldots, m$, of

$$D(x, u(x), \eta_1, \ldots, \eta_{n-1}, \kappa) = 0$$

for all $(\eta_1, \ldots, \eta_{n-1})$, where

$$D(x, u(x), \eta_1, \ldots, \eta_{n-1}, \kappa) = C(x, u(x), x, Q\eta).$$

(ii) System (3.14) is *parabolic* if there exists a regular linear mapping $\zeta = Q\eta$ such that D is independent of κ, that is, D depends on less than n parameters.

(iii) System (3.14) is *elliptic* if $C(x, u, \zeta) = 0$ only if $\zeta = 0$.

Remark. In the elliptic case all derivatives follow from the given data and the given equation.

3.3.1 Examples

1. Beltrami equations

$$
\begin{aligned}
Wu_x - bv_x - cv_y &= 0 &\quad (3.15)\\
Wu_y + av_x + bv_y &= 0, &\quad (3.16)
\end{aligned}
$$

where $W,\ a,\ b,\ c$ are given functions depending of (x, y), $W \neq 0$ and the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is positive definite.

The Beltrami system is a generalization of Cauchy-Riemann equations. The function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, is called a *quasiconform mapping*, see for example [7], Chapter 12 for an application to partial differential equations.

Set

$$A^1 = \begin{pmatrix} W & -b \\ 0 & a \end{pmatrix},\quad A^2 = \begin{pmatrix} 0 & -c \\ W & b \end{pmatrix}.$$

Then the system (3.15), (3.16) can be written as

$$A^1 \begin{pmatrix} u_x \\ v_x \end{pmatrix} + A^2 \begin{pmatrix} u_y \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus,

$$C(x, y, \zeta) = \begin{vmatrix} W\zeta_1 & -b\zeta_1 - c\zeta_2 \\ W\zeta_2 & a\zeta_1 + b\zeta_2 \end{vmatrix} = W(a\zeta_1^2 + 2b\zeta_1\zeta_2 + c\zeta_2^2),$$

which is different from zero if $\zeta \neq 0$ according to the above assumptions. That is, the *Beltrami system is elliptic.*

2. Maxwell equations

The Maxwell equations in the isotropic case are

$$c \text{ rot}_x\ H = \lambda E + \epsilon E_t \tag{3.17}$$

$$c \text{ rot}_x\ E = -\mu H_t, \tag{3.18}$$

where
$E = (e_1, e_2, e_3)^T$ electric field strength, $e_i = e_i(x,t)$, $x = (x_1, x_2, x_3)$,
$H = (h_1, h_2, h_3)^T$ magnetic field strength, $h_i = h_i(x,t)$,
c speed of light,
λ specific conductivity,
ϵ dielectricity constant,
μ magnetic permeability.
Here $c,\ \lambda,\ \epsilon$ and μ are all positive constants.

Set $p_0 = \chi_t,\ p_i = \chi_{x_i}$, $i = 1, \ldots 3$, then the characteristic differential equation is

$$\begin{vmatrix} \epsilon p_0/c & 0 & 0 & 0 & p_3 & -p_2 \\ 0 & \epsilon p_0/c & 0 & -p_3 & 0 & p_1 \\ 0 & 0 & \epsilon p_0/c & p_2 & -p_1 & 0 \\ 0 & -p_3 & p_2 & \mu p_0/c & 0 & 0 \\ p_3 & 0 & -p_1 & 0 & \mu p_0/c & 0 \\ -p_2 & p_1 & 0 & 0 & 0 & \mu p_0/c \end{vmatrix} = 0.$$

The following manipulations lead the a simplification of this equation:
(i) multiply the first three columns with $\mu p_0/c$,
(ii) multiply the 5th column with $-p_3$ and the the 6th column with p_2 and add the sum to the 1st column,
(iii) multiply the 4th column with p_3 and the 6th column with $-p_1$ and add the sum to the 2th column,
(iv) multiply the 4th column with $-p_2$ and the 5th column with p_1 and add the sum to the 3th column,
(v) expand the resulting determinant with respect to the elements of the 6th, 5th and 4th row.

Thus

$$\begin{vmatrix} q+p_1^2 & p_1p_2 & p_1p_3 \\ p_1p_2 & q+p_2^2 & p_2p_3 \\ p_1p_3 & p_2p_3 & q+p_3^2 \end{vmatrix} = 0,$$

where

$$q := \frac{\epsilon\mu}{c^2}p_0^2 - g^2$$

with $g^2 := p_1^2 + p_2^2 + p_3^2$. The evaluation of the above equation leads to $q^2(q+g^2) = 0$, that is,

$$\chi_t^2\left(\frac{\epsilon\mu}{c^2}\chi_t^2 - |\nabla_x\chi|^2\right) = 0.$$

It follows immediately that *Maxwell equations* are a *hyperbolic system*, see an exercise. There are two solutions of this characteristic equation. The first one are characteristic surfaces $\mathcal{S}(t)$, defind by $\chi(x,t) = 0$, which satisfy $\chi_t = 0$. These surfaces are called *stationary waves*. The second type of characteristic surfaces are defined by solutions of

$$\frac{\epsilon\mu}{c^2}\chi_t^2 = |\nabla_x\chi|^2.$$

Functions defined by $\chi = f(n\cdot x - Vt)$ are solutions of this equation. Here is $f(s)$ an arbitrary function with $f' \neq 0$, n is unit vector and $V = c/\sqrt{\epsilon\mu}$. The associated characteristic surfaces $\mathcal{S}(t)$ are defined by

$$\chi(x,t) \equiv f(n\cdot x - Vt) = 0,$$

here we assume that 0 is in he range of $f:\ \mathbb{R} \mapsto \mathbb{R}$. Thus, $\mathcal{S}(t)$ is defined by $n\cdot x - Vt = c$, where c is a fixed constant. It follows that the planes $\mathcal{S}(t)$ with normal n move with speed V in direction of n, see Figure 3.4

V is called *speed* of the plane wave $\mathcal{S}(t)$.

Remark. According to the previous discussions, singularities of a solution of Maxwell equations are located at least on characteristic surfaces.

A special case of Maxwell equations are the **telegraph equations**, which follow from Maxwell equations if div $E = 0$ and div $H = 0$, that is E and H are fields free of source. In fact, it is sufficient to assume that this assumption is satisfied at a fixed time t_0 only, see an exercise.

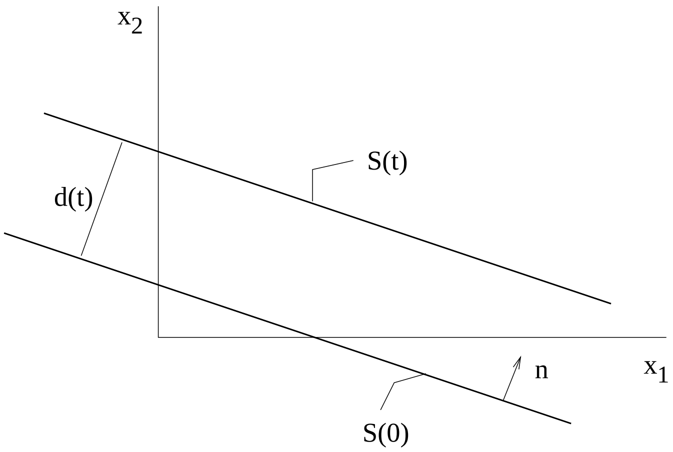

Figure 3.4: $d'(t)$ is the speed of plane waves

Since

$$\mathrm{r}ot_x\ \mathrm{r}ot_x\ A = \mathrm{grad}_x\ \mathrm{d}iv_x\ A - \triangle_x A$$

for each C^2 vector field it follows from Maxwell equations the uncoupled system

$$\begin{aligned} \triangle_x E &= \frac{\epsilon\mu}{c^2}E_{tt} + \frac{\lambda\mu}{c^2}E_t \\ \triangle_x H &= \frac{\epsilon\mu}{c^2}H_{tt} + \frac{\lambda\mu}{c^2}H_t. \end{aligned}$$

3. Equations of gas dynamics

Consider the following two quasilinear equations of first order.

$$v_t + (v \cdot \nabla_x)\ v + \frac{1}{\rho}\nabla_x p = f \quad \text{(Euler equations).}$$

Here is
$v = (v_1, v_2, v_3)$ the vector of speed, $v_i = v_i(x,t)$, $x = (x_1, x_2, x_3)$,
p pressure, $p = (x,t)$,
ρ density, $\rho = \rho(x,t)$,
$f = (f_1, f_2, f_3)$ density of the external force, $f_i = f_i(x,t)$,

$(v \cdot \nabla_x)v \equiv (v \cdot \nabla_x v_1, v \cdot \nabla_x v_2, v \cdot \nabla_x v_3))^T$.

The second equation is

$$\rho_t + v \cdot \nabla_x \rho + \rho \ \mathrm{d}iv_x \ v = 0 \quad \text{(conservation of mass)}.$$

Assume the gas is compressible and that there is a function (state equation)

$$p = p(\rho),$$

where $p(\rho)$ is given such that $p'(\rho) > 0$ if $\rho > 0$. Then the above system of four equtions is

$$\begin{aligned} v_t + (v \cdot \nabla)v + \frac{1}{\rho}p'(\rho)\nabla\rho &= f \qquad (3.19)\\ \rho_t + \rho \ \mathrm{d}iv \ v + v \cdot \nabla\rho &= 0, \qquad (3.20)\end{aligned}$$

where $\nabla \equiv \nabla_x$ and $\mathrm{d}iv \equiv \mathrm{d}iv_x$, that is, these operators apply on the spatial variables only.

The characteristic differential equation is here

$$\begin{vmatrix} \frac{d\chi}{dt} & 0 & 0 & \frac{1}{\rho}p'\chi_{x_1} \\ 0 & \frac{d\chi}{dt} & 0 & \frac{1}{\rho}p'\chi_{x_2} \\ 0 & 0 & \frac{d\chi}{dt} & \frac{1}{\rho}p'\chi_{x_3} \\ \rho\chi_{x_1} & \rho\chi_{x_2} & \rho\chi_{x_3} & \frac{d\chi}{dt} \end{vmatrix} = 0,$$

where

$$\frac{d\chi}{dt} := \chi_t + (\nabla_x\chi) \cdot v.$$

Evaluating the determinant, we get the characteristic differential equation

$$\left(\frac{d\chi}{dt}\right)^2 \left(\left(\frac{d\chi}{dt}\right)^2 - p'(\rho)|\nabla_x\chi|^2\right) = 0. \qquad (3.21)$$

This equation implies consequences for the speed of the move of characteristic surfaces as the following consideration shows.

Consider a family $\mathcal{S}(t)$ of surfaces in $\mathbb{R}^3$ defined by $\chi(x,t) = c$, where $x \in \mathbb{R}^3$ and c is a fixed constant. As usually, we assume that $\nabla_x\chi \neq 0$. One of the two normals on $\mathcal{S}(t)$ at a point of the surface $\mathcal{S}(t)$ is given by, see an exercise,

$$\mathbf{n} = \frac{\nabla_x \chi}{|\nabla_x \chi|}. \tag{3.22}$$

Let $Q_0 \in \mathcal{S}(t_0)$ and let $Q_1 \in \mathcal{S}(t_1)$ be a point on the line defined by $Q_0 + s\mathbf{n}$, where $\mathbf{n}$ is the normal (3.22 on $\mathcal{S}(t_0)$ at Q_0 and $t_0 < t_1$, $t_1 - t_0$ small, see Figure 3.5.

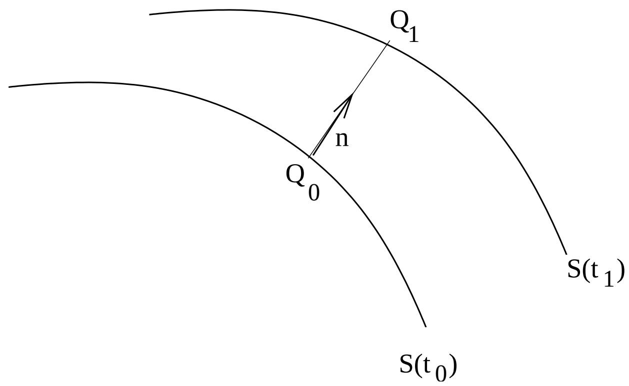

Figure 3.5: Figure to the definition of the speed of a surface

Definition. The limit

$$P = \lim_{t_1 \to t_0} \frac{|Q_1 - Q_0|}{t_1 - t_0}$$

is called *speed* of the surface $\mathcal{S}(t)$.

Proposition. *The speed of the surface $\mathcal{S}(t)$ is*

$$P = -\frac{\chi_t}{|\nabla_x \chi|}. \tag{3.23}$$

Proof. The proof follows from $\chi(Q_0, t_0) = 0$ and $\chi(Q_0 + d\mathbf{n}, t_0 + \triangle t) = 0$, where $d = |Q_1 - Q_0|$ and $\triangle t = t_1 - t_0$. $\Box$

Set $v_n := v \cdot \mathbf{n}$ which is the component of the velocity vector in direction $\mathbf{n}$. From (3.22) it follows

$$v_n = \frac{1}{|\nabla_x \chi|} v \cdot \nabla_x \chi.$$

Definition. $V := P - v_n$, the difference of the speed of the surface and the speed of liquid particles, is called *relative speed.*

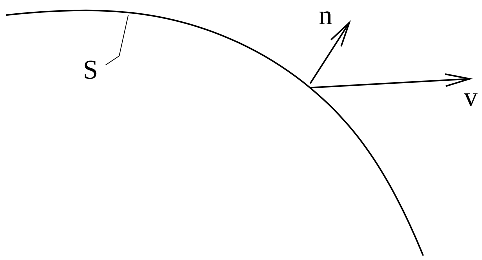

Figure 3.6: Figure to the definition of relative speed

Using the above formulae for P and v_n it follows

$$V = P - v_n = -\frac{\chi_t}{|\nabla_x \chi|} - \frac{v \cdot \nabla_x \chi}{|\nabla_x \chi|} = -\frac{1}{|\nabla_x \chi|}\frac{d\chi}{dt}.$$

Then, it follows from the characteristic equation (3.21) that

$$V^2|\nabla_x \chi|^2 \left(V^2|\nabla_x \chi|^2 - p'(\rho)|\nabla_x \chi|^2\right) = 0.$$

An interesting conclusion is that there are two relative speeds: $V = 0$ or $V^2 = p'(\rho)$.

Definition. $\sqrt{p'(\rho)}$ is called *sound speed.*

3.4 Systems of second order

Here we consider the system

$$\sum_{k,l=1}^{n} A^{kl}(x,u,\nabla u)u_{x_k x_l} + \text{lower order terms} = 0, \tag{3.24}$$

where A^{kl} are $(m \times m)$ matrices and $u = (u_1, \ldots, u_m)^T$. We assume $A^{kl} = A^{lk}$, which is no restriction of generality provided $u \in C^2$ is satisfied. As in the previous sections, the classification follows from the question whether or not we can calculate formally the solution from the differential equations if sufficiently many data are given on an initial manifold. Let the initial manifold $\mathcal{S}$ be given by $\chi(x) = 0$ and assume that $\nabla\chi \neq 0$. The mapping $x = x(\lambda)$, see previous sections, leads to

$$\sum_{k,l=1}^{n} A^{kl}\chi_{x_k}\chi_{x_l}v_{\lambda_n\lambda_n} = \text{terms known on } \mathcal{S},$$

where $v(\lambda) = u(x(\lambda))$.

The characteristic equation is here

$$\det\left(\sum_{k,l=1}^{n} A^{kl}\chi_{x_k}\chi_{x_l}\right) = 0.$$

If there is a solution χ with $\nabla\chi \neq 0$, then it is possible that second derivatives are not continuous in a neighbourhood of $\mathcal{S}$.

Definition. The system is called *elliptic* if

$$\det\left(\sum_{k,l=1}^{n} A^{kl}\zeta_k\zeta_l\right) \neq 0$$

for all $\zeta \in \mathbb{R}^n$, $\zeta \neq 0$.

3.4.1 Examples

1. Navier-Stokes equations

The Navier-Stokes system for a viscous incompressible liquid is

$$\begin{aligned} v_t + (v \cdot \nabla_x)v &= -\frac{1}{\rho}\nabla_x p + \gamma\triangle_x v \\ \mathrm{d}iv_x\ v &= 0, \end{aligned}$$

where ρ is the (constant and positive) density of liquid,
γ is the (constant and positive) viscosity of liquid,
$v = v(x,t)$ velocity vector of liquid particles, $x \in \mathbb{R}^3$ or in $\mathbb{R}^2$,
$p = p(x,t)$ pressure.

The problem is to find solutions $v,\ p$ of the above system.

2. Linear elasticity

Consider the system

$$\rho\frac{\partial^2 u}{\partial t^2} = \mu\triangle_x u + (\lambda+\mu)\nabla_x(\text{d}iv_x\ u) + f. \tag{3.25}$$

Here is, in the case of an elastic body in $\mathbb{R}^3$,
$u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ displacement vector,
$f(x,t)$ density of external force,
ρ (constant) density,
$\lambda,\ \mu$ (positive) Lamé constants.

The characteristic equation is $\det C = 0$, where the elements of the matrix C are given by

$$c_{ij} = (\lambda+\mu)\chi_{x_i}\chi_{x_j} + \delta_{ij}\left(\mu|\nabla_x\chi|^2 - \rho\chi_t^2\right).$$

Thus, the characteristic equation is

$$\left((\lambda+2\mu)|\nabla_x\chi|^2 - \rho\chi_t^2\right)\left(\mu|\nabla_x\chi|^2 - \rho\chi_t^2\right)^2 = 0.$$

It follows that two different speeds of characteristic surfaces $\mathcal{S}(t)$ defined by $\chi(x,t) = const.$ are possible, namely

$$P_1 = \sqrt{\frac{\lambda+2\mu}{\rho}}, \quad \text{and} \quad P_2 = \sqrt{\frac{\mu}{\rho}}.$$

We recall that $P = -\chi_t/|\nabla_x\chi|$.

3.5 Theorem of Cauchy and Kovalevski

Consider the quasilinear system of first order (3.14) of Section 3.3. Assume an initial manifols $\mathcal{S}$ is given by $\chi(x) = 0$, $\nabla\chi \neq 0$ and suppose that χ is not characteristic. Then, see Section 3.3, the system (3.14) can be written as, where we denote λ by x again, and the function $v(\lambda)$ by $u(x)$,

$$u_{x_n} = \sum_{i=1}^{n-1} a^i(x,u)u_{x_i} + b(x,u) \tag{3.26}$$

$$u(x_1,\ldots,x_{n-1},0) = f(x_1,\ldots,x_{n-1}) \tag{3.27}$$

Here is $u = (u_1, \ldots, u_m)^T$, $b = (b_1, \ldots, b_n)^T$ and a^i are $(m \times m)$ matrices. We assume a^i, b and f are in C^∞ with respect to their arguments. From (3.26) and (3.27) it follows that we can calculate formal all derivatives $D^\alpha u$ in in a neigbourhood of the plane $\{x : \ x_n = 0\}$, in particular in a neighbourhood of $0 \in \mathbb{R}^n$. Thus, we have a formally power series of $u(x)$ in $x = 0$:

$$u(x) \sim \sum \frac{1}{\alpha!} D^\alpha u(0) x^\alpha.$$

For notations and definitions used here and in the following see the appendix to this section.

Than, as usually, two questions arise:

(i) Does the power series converge in a neighbourhood of $0 \in \mathbb{R}^n$?

(ii) Is a convergent power series a solution of the initial value problem (3.26), (3.27)?

Remark. Quite different to this power series method is the method of *asymptotic expansions*. Here one is interested in a good approximation of an unknown solution of an equation by a finite sum $\sum_{i=0}^N \phi_i(x)$ of functions ϕ_i. In general, the infinite sum $\sum_{i=0}^\infty \phi_i(x)$ does not converge, in contrast to the power series method of this section, see[12] for asymtotic formulae in capillarity.

Theorem. *There is a neighbourhood of* $0 \in \mathbb{R}^n$ *such there is a real analytic solution of the initial value problem (3.26), (3.27). This solution is unique in the class of real analytic functions.*

Proof. The proof is taken from F. John [8]. We introduce $u - f$ as the new solution for which we are looking at and we add a new coordinate $u^\star$ to the solution vector by setting $u^\star(x) = x_n$. Then

$$u^\star_{x_n} = 1, \ u^\star_{x_k} = 0, \ k = 1, \ldots, n-1, \ u^\star(x_1, \ldots, x_{n-1}, 0) = 0.$$

Then the extended system (3.26), (3.27) is

$$\begin{pmatrix} u_{1,x_n} \\ \vdots \\ u_{m,x_n} \\ u^\star_{x_n} \end{pmatrix} = \sum_{i=1}^{n-1} \begin{pmatrix} a^i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_{1,x_i} \\ \vdots \\ u_{m,x_i} \\ u^\star_{x_i} \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_m \\ 1 \end{pmatrix}$$

and the associated initial condition is $u(x_1,\ldots,x_{n-1},0)=0$. The new u is $u=(u_1,\ldots,u_m)^T$, the new a^i are $a^i(x_1,\ldots,x_{n-1},u_1,\ldots,u_m,u^\star)$ and the new b is $b=(x_1,\ldots,x_{n-1},u_1,\ldots,u_m,u^\star)^T$.

Thus, we are led to an initial value problem of type

$$u_{j,x_n} = \sum_{i=1}^{n-1}\sum_{k=1}^{N} a^i_{jk}(z)u_{k,x_i} + b_j(z),\ \ j=1,\ldots,N \tag{3.28}$$

$$u_j(x) = 0 \ \text{ if } x_n = 0, \tag{3.29}$$

where $j=1,\ldots,N$ and $z=(x_1,\ldots,x_{n-1},u_1,\ldots,u_N)$.

The point is here that a^i_{jk} and b_j are independent of x_n. This fact simplifies the proof of the theorem.

From (3.28) and (3.29) we can calculate formally all $D^\beta u_j$. Thus, we have formal power series for u_j:

$$u_j(x) \sim \sum_\alpha c^{(j)}_\alpha x^\alpha,$$

where

$$c^{(j)}_\alpha = \frac{1}{\alpha!}D^\alpha u_j(0).$$

We will show that these power series are (absolutely) convergent in a neighbourhood of $0\in\mathbb{R}^n$, that is, they are real analytic functions, see the appendix for the definition of real analytic functions. Inserting these fuctions on the left and into the right hand side of (3.28) we obtain on the right and on the left hand side real analytic functions. This follows since compositions of real analytic functions are real analytic again, see Proposition A7 in the appendix to this section. Since the resulting power series on the left and on the rigt have the same coefficients caused by the calculation of the derivatives $D^\alpha u_j(0)$ from (3.28). It follows that $u_j(x)$, $j=1,\ldots,n$, defined by its formal power series are solutions of the initial value problem (3.28), (3.29).

Set
$$d=\left(\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_{N+n-1}}\right)$$

Lemma 1. *Assume* $u\in C^\infty$ *in a neighbourhood of* $0\in\mathbb{R}^n$. *Then*
$$D^\alpha u_j(0)=P_\alpha\left(d^\beta a^i_{jk}(0),d^\gamma b_j(0)\right),$$
where $|\beta|,\ |\gamma|\le|\alpha|$ *and* P_α *are polynomials in the indicated arguments with* **nonnegative** *integers as coefficients which are* **independent** *of* a^i *and of* b.

Proof. It follows from equation (3.28) that
$$D_nD^\alpha u_j(0)=P_\alpha(d^\beta a^i_{jk}(0),d^\gamma b_j(0),D^\delta u_k(0)). \tag{3.30}$$
Here is $\partial/\partial x_n$ and $\beta,\ \gamma,\ \delta$ satisfy the inequalities
$$|\beta|,\ |\gamma|\le|\alpha|,\quad |\delta|\le|\alpha|+1,$$
and, which is essential in the proof, the last coordinates in the multiindices $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\delta=(\delta_1,\ldots,\delta_n)$ satisfy $\delta_n\le\alpha_n$ since the right hand side of (3.28) is independent of x_n. Moreover, it follows from (3.28) that the polynomials P_α have integers as coefficients. The initial condition (3.29) implies
$$D^\alpha u_j(0)=0, \tag{3.31}$$
where $\alpha=(\alpha_1,\ldots,\alpha_{n-1},0)$, that is, $\alpha_n=0$. Then, the proof is by induction with respect to α_n. The induction starts with $\alpha_n=0$, then we replace $D^\delta u_k(0)$ in the right hand side of (3.30) by (3.31), that is by zero. Then it follows from (3.30)
$$D^\alpha u_j(0)=P_\alpha(d^\beta a^i_{jk}(0),d^\gamma b_j(0),D^\delta u_k(0)),$$
where $\alpha=(\alpha_1,\ldots,\alpha_{n-1},1)$. □

Definition. Let $f=(f_1,\ldots,f_m)$, $F=(F_1,\ldots,F_m)$, $f_i=f_i(x)$, $F_i=F_i(x)$, and $f,\ F\in C^\infty$. We say f is *majorized* by F if
$$|D^\alpha f_k(0)|\le D^\alpha F_k(0),\quad k=1,\ldots,m$$
for all α. We write $f<<F$, if f is majorized by F.

Definition. The initial value problem

$$U_{j,x_n} = \sum_{i=1}^{n-1}\sum_{k=1}^{N} A^i_{jk}(z)U_{k,x_i} + B_j(z) \tag{3.32}$$
$$U_j(x) = 0 \text{ if } x_n = 0, \tag{3.33}$$

$j = 1, \ldots, N$, A^i_{jk}, B_j real analytic, ist called *majorizing problem* to (3.28), (3.29) if

$$a^i_{jk} << A^i_{jk} \text{ and } b_j << B_j.$$

Lemma 2. *The formal power series*

$$\sum_\alpha \frac{1}{\alpha!} D^\alpha u_j(0) x^\alpha,$$

where $D^\alpha u_j(0)$ are defined in Lemma 1, is convergent in a neighbourhood of $0 \in \mathbb{R}^n$ if there exists a majorizing problem which has a real analytic solution U in $x = 0$, and

$$|D^\alpha u_j(0)| \leq D^\alpha U_j(0).$$

Proof. It follows from Lemma 1 and from the assumption of this lemma that

$$\begin{aligned} |D^\alpha u_j(0)| &\leq P_\alpha\left(|d^\beta a^i_{jk}(0)|, |d^\gamma b_j(0)|\right) \\ &\leq P_\alpha\left(|d^\beta A^i_{jk}(0)|, |d^\gamma B_j(0)|\right) \equiv D^\alpha U_j(0). \end{aligned}$$

The formal power series

$$\sum_\alpha \frac{1}{\alpha!} D^\alpha u_j(0) x^\alpha,$$

is convergent since

$$\sum_\alpha \frac{1}{\alpha!} |D^\alpha u_j(0) x^\alpha| \leq \sum_\alpha \frac{1}{\alpha!} D^\alpha U_j(0) |x^\alpha|.$$

The right hand side is convergent in a neighbourhood of $x \in \mathbb{R}^n$ by assumption. $\square$

Lemma 3. *There is a majorising problem which has a real analytic solution.*

Proof. Since $a^i_{ij}(z)$, $b_j(z)$ are real analytic in a neighbourhood of $z=0$ it follows from Proposition A5 in the appendix of this section that there are positive constants M and r such that all these functions are majorized by

$$\frac{Mr}{r-z_1-\ldots-z_{N+n-1}}.$$

Thus, a majorizing problem is

$$\begin{aligned} U_{j,x_n} &= \frac{Mr}{r-x_1-\ldots-x_{n-1}-U_1-\ldots-U_N}\left(1+\sum_{i=1}^{n-1}\sum_{k=1}^{N}U_{k,x_i}\right)\\ U_j(x) &= 0 \ \text{ if } x_n=0, \end{aligned}$$

$j=1,\ldots,N$.

The solution of this problem is

$$U_j(x_1,\ldots,x_{n-1},x_n)=V(x_1+\ldots+x_{n-1},x_n),\ j=1,\ldots,N,$$

where $V(s,t)$, $s=x_1+\ldots+x_{n-1}$, $t=x_n$ is the solution of the Cauchy initial value problem

$$\begin{aligned} V_t &= \frac{Mr}{r-s-NV}\left(1+N(n-1)V_s\right)\\ V(s,0) &= 0. \end{aligned}$$

The solution is, see an exercise,

$$V(s,t)=\frac{1}{Nn}\left(r-s-\sqrt{(r-s)^2-2nMNrt}\right).$$

This function is real analytic in (s,t) at $(0,0)$. It follows that $U_j(x)$ are also real analytic functions. Thus the Cauchy-Kovalevski theorem is shown. □

Examples:

1. Ordinary differential equations

Consider the initial value problem

$$\begin{aligned} y'(x) &= f(x,y(x))\\ y(x_0) &= y_0, \end{aligned}$$

where $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}^n$ are given. Assume $f(x, y)$ is real analytic in a neighbourhood of $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}^n$. Then it follows from the above theorem that there exists an analytic solution $y(x)$ of the initial value problem in a neighbourhood of x_0. This solution is unique in the class of analytic functions according to the theorem of Cauchy-Kovalevski. From the Picard-Lindelöf theorem it follows that this analytic solution is even unique in the class of C^1-functions.

2. Partial differential equations of second order

Consider the boundary value problem for two variables

$$\begin{aligned} u_{yy} &= f(x, y, u, u_x, u_y, u_{xx}, u_{xy}) \\ u(x, 0) &= \phi(x) \\ u_y(x, 0) &= \psi(x). \end{aligned}$$

We assume that ϕ, ψ are analytic in a neighbourhood of $x = 0$ and that f is real analytic in a neighbourhood of

$$(0, 0, \phi(0), \phi'(0), \psi(0), \psi'(0)).$$

There exists a real analytic solution in a neigbourhood of $0 \in \mathbb{R}^2$ *of the above initial value problem.*

In particular, there is a real analytic solution in a neigbourhood of $0 \in \mathbb{R}^2$ of the initial value problem

$$\begin{aligned} \triangle u &= 1 \\ u(x, 0) &= 0 \\ u_y(x, 0) &= 0. \end{aligned}$$

The proof follows by writing the above problem as a system. Set $p = u_x$, $q = u_y$, $r = u_{xx}$, $s = u_{xy}$, $t = u_{yy}$, then

$$t = f(x, y, u, p, q, r, s).$$

Set $U=(u,p,q,r,s,t)^T$, $b=(q,0,t,0,0,f_y+f_uq+f_qt)^T$ and

$$A=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & f_p & 0 & f_r & f_s \end{pmatrix}.$$

Then the rewritten differential equation is the system $U_y=AU_x+b$ with the initial condition

$$U(x,0)=(\phi(x),\phi'(x),\psi(x),\phi''(x),\psi'(x),f_0(x))\,,$$

where $f_0(x)=f(x,0,\phi(x),\phi'(x),\psi(x),\phi''(x),\psi'(x))$.

3.5.1 Appendix: Real analytic functions

Multi-index notation

The following multi-index notation simplifies many presentations of formulae. Let $x=(x_1,\ldots,x_n)$ and

$$u:\ \Omega\subset\mathbb{R}^n\mapsto\mathbb{R}\ \ \text{(or }\mathbb{R}^m\text{ for systems)}.$$

The n-tupel of nonnegative integers (including zero)

$$\alpha=(\alpha_1,\ldots,\alpha_n)$$

is called *multi-index*. Set

$$\begin{aligned}
|\alpha| &= \alpha_1+\ldots+\alpha_n \\
\alpha! &= \alpha_1!\alpha_2!\cdot\ldots\cdot\alpha_n! \\
x^\alpha &= x_1^{\alpha_1}x_2^{\alpha_2}\cdot\ldots\cdot x_n^{\alpha_n}\ \ \text{(for a monom)} \\
D_k &= \frac{\partial}{\partial x_k} \\
D &= (D_1,\ldots,D_n) \\
Du &= (D_1u,\ldots,D_nu)\equiv\nabla u\equiv grad\ u \\
D^\alpha &= D_1^{\alpha_1}D_2^{\alpha_2}\cdot\ldots\cdot D_n^{\alpha_n}\equiv\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_n^{\alpha_n}}.
\end{aligned}$$

Define a partial order by

$$\alpha \geq \beta \text{ if and only if } \alpha_i \geq \beta_i \text{ for all } i.$$

Sometimes we use the notations

$$\mathbf{0} = (0, 0 \ldots, 0), \quad \mathbf{1} = (1, 1 \ldots, 1),$$

where $\mathbf{0}$, $\mathbf{1} \in \mathbb{R}^n$.

Using this multi-index notion, we have

1.

$$(x+y)^\alpha = \sum_{\substack{\beta,\gamma \\ \beta+\gamma=\alpha}} \frac{\alpha!}{\beta!\gamma!} x^\beta y^\gamma,$$

where $x\ y \in \mathbb{R}^n$ and α, β, γ are multi-indices.

2. Taylor expansion for a *polynomial* $f(x)$ of degree m:

$$f(x) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \left(D^\alpha f(0)\right) x^\alpha,$$

here is $D^\alpha f(0) = (D^\alpha f(x))\,|_{x=0}$.

3. Let $x = (x_1, \ldots, x_n)$ and $m \geq 0$ an integer, then

$$(x_1 + \ldots + x_n)^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha.$$

4.

$$\alpha! \leq |\alpha|! \leq n^{|\alpha|}\alpha!.$$

5. Leibniz's rule:

$$D^\alpha(fg) = \sum_{\substack{\beta,\gamma \\ \beta+\gamma=\alpha}} \frac{\alpha!}{\beta!\gamma!} (D^\beta f)(D^\gamma g).$$

6.

$$\begin{aligned} D^\beta x^\alpha &= \frac{\alpha!}{(\alpha-\beta)!} x^{\alpha-\beta} \ \text{ if } \alpha \geq \beta \\ D^\beta x^\alpha &= 0 \ \text{ otherwise.} \end{aligned}$$

7. Directional derivative:

$$\frac{d^m}{dt^m} f(x+ty) = \sum_{|\alpha|=m} \frac{|\alpha|!}{\alpha!} \left(D^\alpha f(x+ty)\right) y^\alpha,$$

where $y, \ y \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

8. Taylor's theorem: Let $u \in C^{m+1}$ in a neighbourhood $U(y)$ of y, then, if $x \in U(y)$,

$$u(x) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \left(D^\alpha u(y)\right)(x-y)^\alpha + R_m,$$

where

$$R_m = \sum_{|\alpha|=m+1} \frac{1}{\alpha!} \left(D^\alpha u(y+\delta(x-y))\right) x^\alpha, \ 0 < \delta < 1,$$

$\delta = \delta(u, m, x, y)$, or

$$R_m = \frac{1}{m!} \int_0^1 (1-t)^m \Phi^{(m+1)}(t) \ dt,$$

where $\Phi(t) = u(y + t(x-y))$. It follows from 7. that

$$R_m = (m+1) \sum_{|\alpha|=m+1} \frac{1}{\alpha!} \left(\int_0^1 (1-t) D^\alpha u(y+t(x-y)) \ dt \right) (x-y)^\alpha.$$

9. Using multi-index notation, the general linear partial differential equation of order m can be written as

$$\sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u = f(x) \ \text{ in } \Omega \subset \mathbb{R}^n.$$

Power series

Here we collect some definitions and results for power series in $\mathbb{R}^n$.

Definition. Let $c_\alpha \in \mathbb{R}$ (or $\in \mathbb{R}^m$). The series

$$\sum_\alpha c_\alpha \equiv \sum_{m=0}^\infty \left(\sum_{|\alpha|=m} c_\alpha \right)$$

is said to be convergent if

$$\sum_\alpha |c_\alpha| \equiv \sum_{m=0}^\infty \left(\sum_{|\alpha|=m} |c_\alpha| \right)$$

is convergent.

Remark. According to the above definition, a convergent series is absolutely convergent. Then, it follows that we can rearrange the order of summation.

Using above multi-index notation and keeping in mind that we can rearrange convergent series, we have

10. Let $x \in \mathbb{R}^n$, then

$$\begin{aligned} \sum_\alpha x^\alpha &= \prod_{i=1}^n \left(\sum_{\alpha_i=0}^\infty x_i^{\alpha_i} \right) \\ &= \frac{1}{(1-x_1)(1-x_2)\cdot \ldots \cdot (1-x_n)} \\ &= \frac{1}{(\mathbf{1}-x)^{\mathbf{1}}}, \end{aligned}$$

provided $|x_i| < 1$ is satisfied for each i. This follows since we have in the first line the same terms on the left and on the right hand side.

11. Assume $x \in \mathbb{R}^n$ and $|x_1| + |x_2| + \ldots + |x_n| < 1$, then

$$\begin{aligned}\sum_\alpha \frac{|\alpha|!}{\alpha!} x^\alpha &= \sum_{j=0}^\infty \sum_{|\alpha|=j} \frac{|\alpha|!}{\alpha!} x^\alpha \\ &= \sum_{j=0}^\infty (x_1 + \ldots + x_n)^j \\ &= \frac{1}{1 - (x_1 + \ldots + x_n)}.\end{aligned}$$

12. Let $x \in \mathbb{R}^n$, $|x_i| < 1$ for all i, and β is a given multi-index. Then

$$\begin{aligned}\sum_{\alpha \geq \beta} \frac{\alpha!}{(\alpha - \beta)!} x^{\alpha - \beta} &= D^\beta \frac{1}{(\mathbf{1} - x)^{\mathbf{1}}} \\ &= \frac{\beta!}{(\mathbf{1} - x)^{\mathbf{1} + \beta}} .\end{aligned}$$

13. Let $x \in \mathbb{R}^n$ and $|x_1| + \ldots + |x_n| < 1$. Then

$$\begin{aligned}\sum_{\alpha \geq \beta} \frac{|\alpha|!}{(\alpha - \beta)!} x^{\alpha - \beta} &= D^\beta \frac{1}{1 - x_1 - \ldots - x_n} \\ &= \frac{|\beta|!}{(1 - x_1 - \ldots - x_n)^{1 + |\beta|}} .\end{aligned}$$

Consider the power series

$$\sum_\alpha c_\alpha x^\alpha \tag{3.34}$$

and assume this series is convergent for a $z \in \mathbb{R}^n$. Then, by definition,

$$\mu := \sum_\alpha |c_\alpha||z^\alpha| < \infty$$

and the series (3.34) is uniformly convergent for all $x \in Q(z)$, where

$$Q(z) : \quad |x_i| \leq |z_i| \quad \text{for all} \quad i.$$

Thus, the power series (3.34) defines a continuous function defined on $Q(z)$,

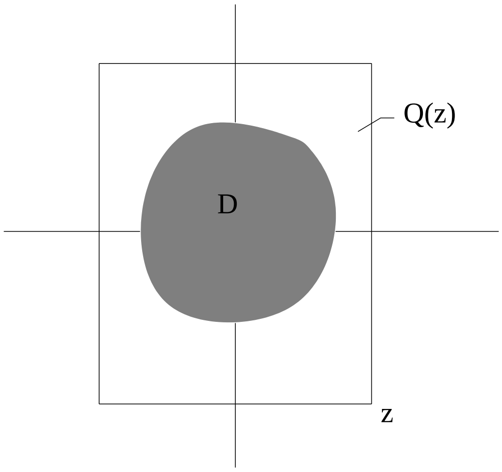

Figure 3.7: Definition of $D \in Q(z)$

according to a theorem of Weierstraß.

The interior of $Q(z)$ is not empty if and only if $z_i \neq 0$ for all i, see Figure 3.7. For x in a fixed compact subset D of $Q(z)$ there is a q, $0 < q < 1$, such that

$$|x_i| \leq q|z_i| \quad \text{for all } i.$$

Set

$$f(x) = \sum_\alpha c_\alpha x^\alpha.$$

Proposition A1. (i) *In every compact subset D of $Q(z)$ one has $f \in C^\infty(D)$ and the formal differentiate series, that is $\sum_\alpha D^\beta c_\alpha x^\alpha$, is uniformly convergent on the closure of D and is equal to $D^\beta f$.*

(ii)

$$|D^\beta f(x)| \leq M|\beta|!r^{-|\beta|} \quad \textit{in } D,$$

where

$$M = \frac{\mu}{(1-q)^n}, \qquad\qquad r = (1-q)\min_i |z_i|.$$

Proof. See F. John [8], p. 64. Or an exercise. Hint: Use formula **12.** where x is replaced by $(q, \ldots, q)$.

Remark. From the proposition it follows

$$c_\alpha = \frac{1}{\alpha!} D^\alpha f(0).$$

Definition. Assume f is defined on a domain $\Omega \subset \mathbb{R}^n$, then, f is said to be *real analytic in* $y \in \Omega$ if there are $c_\alpha \in \mathbb{R}$ and if there is a neighbourhoud $N(y)$ of y such that

$$f(x) = \sum_\alpha c_\alpha (x-y)^\alpha$$

for all $x \in N(y)$, and the series converges (absolutely) for each $x \in N(y)$.
A function f is called real analytic in Ω if it is real analytic for each $y \in \Omega$.
We will write $f \in C^\omega(\Omega)$ in the case that f is real analytic in the domain Ω.
A vector valued function $f(x) = (f_1(x), \ldots, f_m)$ is called real analytic if each coordinate is real analytic.

Proposition A2. (i) *Let* $f \in C^\omega(\Omega)$. *Then* $f \in C^\infty(\Omega)$.

(ii) *Assume* $f \in C^\omega(\Omega)$. *Then for each* $y \in \Omega$ *there exists a neighbourhood* $N(y)$ *and positive constants* M, r *such that*

$$f(x) = \sum_\alpha \frac{1}{\alpha!} (D^\alpha f(y))(x-y)^\alpha$$

for all $x \in N(y)$, *and the series converges (absolutely) for each* $x \in N(y)$, *and*

$$|D^\beta f(x)| \le M |\beta|! r^{-|\beta|}.$$

The proof follows from Proposition A1.

An open set $\Omega \in \mathbb{R}^n$ is called *connected* if Ω is not a union of two non-empty open sets with empty intersection. From the theory of one complex variable we know that a continuation of an analytic function is uniquely determined. The same is true for real analytic functions.

Proposition A3. Assume $f \in C^\omega(\Omega)$ and Ω is connected. Then f is determined uniquely if for one $z \in \Omega$ all $D^\alpha f(z)$ are known.

Proof. See F. John [8], p. 65. Suppose $g,\ h\in C^\omega(\Omega)$ and $D^\alpha g(z)=D^\alpha h(z)$ for every α. Set $f=g-h$ and

$$\begin{aligned}\Omega_1 &= \{x\in\Omega:\ D^\alpha f(x)=0 \ \text{ for all } \alpha\},\\ \Omega_2 &= \{x\in\Omega:\ D^\alpha f(x)\neq 0 \ \text{ for at least one } \alpha\}.\end{aligned}$$

The set Ω_2 is open since D^α are continuous in Ω. The set Ω_1 is also open since $f(x)=0$ in a neighbourhood of $y\in\Omega_1$. This follows from

$$f(x)=\sum_\alpha \frac{1}{\alpha!}(D^\alpha f(y))(x-y)^\alpha.$$

Since $z\in\Omega_1$, that is $\Omega_1\neq\emptyset$, it follows $\Omega_2=\emptyset$. □

It was shown in Proposition A2 that derivatives of a real analytic function satisfy estimates. On the other hand it follows, see the next proposition, that a function $f\in C^\infty$ is real analytic if these estimates are satisfied.

Definition. Let $y\in\Omega$ and $M,\ r$ positive constants. Then f is said to be in the class $C_{M,r}(y)$ if $f\in C^\infty$ in a neighbourhood of y and if

$$|D^\beta f(y)|\le M|\beta|!r^{-|\beta|}$$

for all β.

Proposition A4. *$f\in C^\omega(\Omega)$ if and only if $f\in C^\infty(\Omega)$ and for every compact subset $S\subset\Omega$ there are positive constants $M,\ r$ such that*

$$f\in C_{M,r}(y)\ \ \textit{for all } y\in S.$$

Proof. See F. John [8], pp. 65-66. We will prove the local version of the proposition, that is, we show it for each fixed $y\in\Omega$. The general version follows from Heine-Borel theorem. Because of Proposition A3 it remains to show that taylor series

$$\sum_\alpha \frac{1}{\alpha!}D^\alpha f(y)(x-y)^\alpha$$

converges (absolutely) in a neighbourhood of y and that this series is equal to $f(x)$.

Define a neighbourhood of y by

$$U_d(y) = \{x \in \Omega : \ \ |x_1 - y_1| + \ldots + |x_n - y_n| < d\},$$

where d is a sufficiently small positive constant. Set $\Phi(t) = f(y + t(x - y))$. The one-dimensional Taylor theorem says

$$f(x) = \Phi(1) = \sum_{k=0}^{j-1} \frac{1}{k!}\Phi^{(k)}(0) + r_j,$$

where

$$r_j = \frac{1}{(j-1)!}\int_0^1 (1-t)^{j-1}\Phi^{(j)}(t)\ dt.$$

From formula **7.** for directional derivatives it follows for $x \in U_d(y)$ that

$$\frac{1}{j!}\frac{d^j}{dt^j}\Phi(t) = \sum_{|\alpha|=j} \frac{1}{\alpha!} D^\alpha f(y + t(x-y))(x-y)^\alpha.$$

From the assumption and the multinomial formula **3.** we get for $0 \le t \le 1$

$$\begin{aligned} \left|\frac{1}{j!}\frac{d^j}{dt^j}\Phi(t)\right| &\le M \sum_{|\alpha|=j} \frac{|\alpha|!}{\alpha!} r^{-|\alpha|}\,|(x-y)^\alpha| \\ &= Mr^{-j}\left(|x_1 - y_1| + \ldots + |x_n - y_n|\right)^j \\ &\le M\left(\frac{d}{r}\right)^j . \end{aligned}$$

Choose $d > 0$ such that $d < r$, then the Taylor series converges (absolutely) in $U_d(y)$ and it is equal to $f(x)$ since the remainder satisfies, see the above estimate,

$$|r_j| = \left|\frac{1}{(j-1)!}\int_0^1 (1-t)^{j-1}\Phi^j(t)\ dt\right| \le M\left(\frac{d}{r}\right)^j .$$

$\Box$

We remember that the notation $f << F$ (f is majorized by F) was defined in the previous section.

Proposition A5. (i) $f = (f_1, \ldots, f_m) \in C_{M,r}(0)$ *if and only if* $f << (\Phi, \ldots, \Phi)$, *where*

$$\Phi(x) = \frac{Mr}{r - x_1 - \ldots - x_n} .$$

(ii) $f \in C_{M,r}(0)$ *and* $f(0) = 0$ *if and only if*

$$f << (\Phi - M, \dots, \Phi - M),$$

where

$$\Phi(x) = \frac{M(x_1 + \ldots + x_n)}{r - x_1 - \ldots - x_n}\ .$$

Proof.

$$D^\alpha \Phi(0) = M|\alpha|!r^{-|\alpha|}.$$

□

Remark. The definition of $f << F$ implies, trivially, that $D^\alpha f << D^\alpha F$.

The next proposition shows that compositions majorize if the involved functions majorize. More precisely, we have

Proposition A6. *Let* $f,\ F:\ \mathbb{R}^n \mapsto \mathbb{R}^m$ *and* $g,\ G$ *maps a neighbourhood of* $0 \in \mathbb{R}^m$ *into* $\mathbb{R}^p$. *Assume all functions* $f(x),\ F(x),\ g(u),\ G(u)$ *are in* C^∞, $f(0) = F(0) = 0$, $f << F$ *and* $g << G$. *Then* $g(f(x)) << G(F(x))$.

Proof. See F. John [8], p. 68. Set

$$h(x) = g(f(x)), \quad H(x) = G(F(x)).$$

For each coordinate h_k of h we have, according to the chain rule,

$$D^\alpha h_k(0) = P_\alpha(\delta^\beta g_l(0), D^\gamma f_j(0)),$$

where P_α are polynomials with *nonnegative* integers as coefficients, P_α are independent on g or f and $\delta := (\partial/\partial u_1, \dots, \partial/\partial u_m)$. Thus,

$$\begin{aligned} |D^\alpha h_k(0)| &\le P_\alpha(|\delta^\beta g_l(0)|, |D^\gamma f_j(0)|) \\ &\le P_\alpha(\delta^\beta G_l(0), D^\gamma F_j(0)) \\ &= D^\alpha H_k(0). \end{aligned}$$

□

Using this result and Proposition A4 which characterizes real analytic functions, it follows that compositions of real analytic functions are real analytic functions again.

Proposition A7. *Assume $f(x)$ and $g(u)$ are real analytic, then $g(f(x))$ is real analytic at all x for which $f(x)$ is in the domain of definition of g.*

Proof. See F. John [8], p. 68. Assume that f maps a neighbourhood of $y \in \mathbb{R}^n$ in $\mathbb{R}^m$ and g maps a neighbourhood of $v = f(y)$ in $\mathbb{R}^m$. Then $f \in C_{M,r}(y)$ and $g \in C_{\mu,\rho}(v)$ implies

$$h(x) := g(f(x)) \in C_{\mu,\rho r/(mM+\rho)}(y).$$

Once one has shown this inclusion, the proposition follows from Proposition A4. To show the inclusion, we set

$$h(y+x) := g(f(y+x)) \equiv g(v + f(y+x) - f(x)) =: g^*(f^*(x)),$$

where $v = f(y)$ and

$$\begin{aligned} g^*(u) : &= g(v+u) \in C_{\mu,\rho}(0) \\ f^*(x) : &= f(y+x) - f(y) \in C_{M,r}(0). \end{aligned}$$

In the above formulae $v,\ y$ are considered as fixed parameters. From Proposition A5 it follows

$$\begin{aligned} f^*(x) &<< (\Phi - M, \ldots, \Phi - M) =: F \\ g^*(u) &<< (\Psi, \ldots, \Psi) =: G, \end{aligned}$$

where

$$\begin{aligned} \Phi(x) &= \frac{Mr}{r - x_1 - x_2 - \ldots - x_n} \\ \Psi(u) &= \frac{\mu\rho}{\rho - x_1 - x_2 - \ldots - x_n}. \end{aligned}$$

From Proposition A6 we get

$$h(y+x) << (\chi(x), \ldots, \chi(x)) \equiv G(F),$$

where

$$\begin{aligned}\chi(x) &= \frac{\mu\rho}{\rho - m(\Phi(x) - M)} \\ &= \frac{\mu\rho(r - x_1 - \ldots - x_n)}{\rho r - (\rho + mM)(x_1 + \ldots + x_n)} \\ &<< \frac{\mu\rho r}{\rho r - (\rho + mM)(x_1 + \ldots + x_n)} \\ &= \frac{\mu\rho r/(\rho + mM)}{\rho r/(\rho + mM) - (x_1 + \ldots x_n)}.\end{aligned}$$

See an exercise for the "<<"-inequality. □

3.6 Exercises

1. Let χ: $\mathbb{R}^n \to \mathbb{R}$ in C^1, $\nabla\chi \neq 0$. Show that for given $x_0 \in \mathbb{R}^n$ there is in a neighbourhood of x_0 a local diffeomorphism $\lambda = \Phi(x)$, Φ : $(x_1, \ldots, x_n) \mapsto (\lambda_1, \ldots, \lambda_n)$, such that $\lambda_n = \chi(x)$.

2. Show that the differential equation
$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + \text{lower order terms} = 0$$
is elliptic if $ac - b^2 > 0$, parabolic if $ac - b^2 = 0$ and hyperbolic if $ac - b^2 < 0$.

3. Show that in the hyperbolic case there exists a solution of $\phi_x + \mu_1\phi_y = 0$, see equation (3.9), such that $\nabla\phi \neq 0$.

   Hint. Consider an appropriate Cauchy initial value problem.

4. Show equation (3.4).

5. Find the type of
$$Lu := 2u_{xx} + 2u_{xy} + 2u_{yy} = 0$$
and transform this equation into an equation with vanishing mixed derivatives by using the orthogonal mapping (transform to principal axis) $x = Uy$, U orthogonal.

6. Determine the type of the following equation at $(x, y) = (1, 1/2)$.
$$Lu := xu_{xx} + 2yu_{xy} + 2xyu_{yy} = 0.$$

7. Find all C^2-solutions of
$$u_{xx} - 4u_{xy} + u_{yy} = 0.$$
Hint. Transform to principal axis and stretching of axis leads to the wave equation.

8. Oscillations of a beam are described by

$$\begin{aligned} w_x - \frac{1}{E}\sigma_t &= 0 \\ \sigma_x - \rho w_t &= 0, \end{aligned}$$

where σ stresses, w deflection of the beam and $E,\ \rho$ positive constants.
a) Determine the type of the system.
b) Transform the system into two uncoupled equations, that is, $w,\ \sigma$ occur only in one equation, respectively.
c) Find non-zero solutions.

9. Find nontrivial solutions ($\nabla\chi \neq 0$) of the characteristic equation to

$$x^2 u_{xx} - u_{yy} = f(x, y, u, \nabla u),$$

where f is given.

10. Determine the type of

$$u_{xx} - x u_{yx} + u_{yy} + 3u_x = 2x,$$

where $u = u(x, y)$.

11. Transform equation

$$u_{xx} + (1 - y^2)u_{xy} = 0,$$

$u = u(x, y)$ into its normal form.

12. Show that

$$\lambda = \frac{1}{(1 + |p|^2)^{3/2}}, \quad \Lambda = \frac{1}{(1 + |p|^2)^{1/2}}.$$

are the minimum and maximum of eigenvalues of the matrix (a_{ij}), where

$$a_{ij} = \left(1 + |p|^2\right)^{-1/2} \left(\delta_{ij} - \frac{p_i p_j}{1 + |p|^2}\right).$$

13. Show that Maxwell equations are a hyperbolic system.

14. Consider Maxwell equations and prove that $\mathrm{d}iv\ E = 0$ and $\mathrm{d}iv\ H = 0$ for all t if these equations are satisfied for a fixed time t_0.

    Hint. $\mathrm{d}iv\ \mathrm{r}ot = 0$.

15. Assume a characteristic surface $\mathcal{S}(t)$ in $\mathbb{R}^3$ is defined by $\chi(x, y, z, t) = const.$ such that $\chi_t = 0$ and $\chi_z \neq 0$. Show that $\mathcal{S}(t)$ has a nonparametric representation $z = u(x, y, t)$ with $u_t = 0$, that is $\mathcal{S}(t)$ is independent of t.

16. Prove formula (3.22) for the normal on a surface.

17. Prove formula (3.23) for the speed of the surface $\mathcal{S}(t)$.

18. Write the Navier-Stokes system as a system of type (3.24).

19. Show that the following system (linear elasticity, stationary case of (3.25) in the two dimensional case) is elliptic

$$\mu \triangle u + (\lambda + \mu)\ \mathrm{grad}(\mathrm{div}\ u) + f = 0,$$

    where $u = (u_1, u_2)$. The vector $f = (f_1, f_2)$ is given and λ, μ are positive constants.

20. Discuss the type of the following system in stationary gas dynamics (isentrop flow) in $\mathbb{R}^2$.

$$\begin{aligned} \rho u u_x + \rho v u_y + a^2 \rho_x &= 0 \\ \rho u v_x + \rho v v_y + a^2 \rho_y &= 0 \\ \rho(u_x + v_y) + u \rho_x + v \rho_y &= 0. \end{aligned}$$

    Here are (u, v) velocity vector, ρ density and $a = \sqrt{p'(\rho)}$ the sound velocity.

21. Show formula 7. (directional derivative) of the lecture notes.

    Hint. Induction with respect to m.

22. Let $y = y(x)$ be the solution of:

$$\begin{aligned} y'(x) &= f(x, y(x)) \\ y(x_0) &= y_0, \end{aligned}$$

where f is real analytic in a neighbourhood of $(x_0, y_0) \in \mathbb{R}^2$. Find the polynomial P of degree 2 such that

$$y(x) = P(x - x_0) + O(|x - x_0|^3)$$

as $x \to x_0$.

23. Let u be the solution of

$$\begin{aligned} \triangle u &= 1 \\ u(x, 0) &= u_y(x, 0) = 0. \end{aligned}$$

Find the polynomial P of degree 2 such that

$$u(x, y) = P(x, y) + O((x^2 + y^2)^{3/2})$$

as $(x, y) \to (0, 0)$.

24. Solve the Cauchy initial value problem

$$\begin{aligned} V_t &= \frac{Mr}{r - s - NV}(1 + N(n-1)V_s) \\ V(s, 0) &= 0. \end{aligned}$$

Hint. Multiply the differential equation with $(r - s - NV)$.

25. Write $\triangle^2 u = -u$ as a system of first order.

Hint. $\triangle^2 u \equiv \triangle(\triangle u)$.

26. Write the minimal surface equation

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}\right) = 0$$

as a system of first order.

Hint. $v_1 := u_x/\sqrt{1 + u_x^2 + u_y^2}$, $v_2 := u_y/\sqrt{1 + u_x^2 + u_y^2}$.

27. Let $f:\ \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^m$ be real analytic in (x_0, y_0). Show that a real analytic solution in a neighbourhood of x_0 of the problem

$$\begin{aligned} y'(x) &= f(x,y) \\ y(x_0) &= y_0 \end{aligned}$$

exists and is equal to the $C^1[x_0-\epsilon, x_0+\epsilon]$-solution, $\epsilon > 0$ sufficiently small.

28. Show (see the proof of Proposition A7)

$$\frac{\mu\rho(r - x_1 - \ldots - x_n)}{\rho r - (\rho + mM)(x_1 + \ldots + x_n)} << \frac{\mu\rho r}{\rho r - (\rho + mM)(x_1 + \ldots + x_n)}.$$

Hint. Leibniz's rule.

29. Let $u(x_1, x_2)$ be a solution of Laplace equation $\triangle u = 0$ such that $u = f(\theta)$, $\frac{\partial u}{\partial r} = g(\theta)$ if $r = 1$. The given Cauchy initial data f, g are real analytic and 2π- periodic. Here r, θ denote polar coordinates, that is, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. Show that u is a real analytic for all θ, and $|r-1|$ sufficiently small.

Chapter 4

Hyperbolic equations

Here we consider hyperbolic equations of second order, mainly wave equations.

4.1 One dimensional wave equation

The one-dimensional wave equation is given by

$$\frac{1}{c^2}u_{tt} - u_{xx} = 0, \tag{4.1}$$

where $u = u(x,t)$ is a scalar function of two variables and c is a positive constant. According to previous considerations, all C^2-solutions of the wave equation are given by

$$u(x,t) = f(x + ct) + g(x - ct), \tag{4.2}$$

where f and g are arbitrary C^2-functions.

The *Cauchy initial value problem for the wave equation* is to find a C^2-solution of

$$\begin{aligned} \frac{1}{c^2}u_{tt} - u_{xx} &= 0 \\ u(x,0) &= \alpha(x) \\ u_t(x,0) &= \beta(x), \end{aligned}$$

where $\alpha,\ \beta \in C^2(-\infty,\infty)$ are given.

Proposition 4.1. *There exists a unique $C^2(\mathbb{R}\times\mathbb{R})$-solution of the Cauchy initial value problem, and this solution is given by the d'Alembert's*[1] *formula*

$$u(x,t)=\frac{\alpha(x+ct)+\alpha(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}\beta(s)\ ds. \tag{4.3}$$

Proof. Assume there is a solution $u(x,t)$ of Cauchy inititial value problem, then it follows from (4.2)

$$\begin{aligned} u(x,0) &= f(x)+g(x)=\alpha(x) &(4.4)\\ u_t(x,0) &= cf'(x)-cg'(x)=\beta(x). &(4.5)\end{aligned}$$

From (4.4) it follows

$$f'(x)+g'(x)=\alpha'(x),$$

which implies, together with (4.5),

$$\begin{aligned} f'(x) &= \frac{\alpha'(x)+\beta(x)/c}{2}\\ g'(x) &= \frac{\alpha'(x)-\beta(x)/c}{2}.\end{aligned}$$

Thus,

$$\begin{aligned} f(x) &= \frac{\alpha(x)}{2}+\frac{1}{2c}\int_0^x\beta(s)\ ds+C_1\\ g(x) &= \frac{\alpha(x)}{2}-\frac{1}{2c}\int_0^x\beta(s)\ ds+C_2.\end{aligned}$$

The constants C_1, C_2 satisfy

$$C_1+C_2=f(x)+g(x)-\alpha(x)=0,$$

see (4.4). Thus each C^2-solution of the Cauchy initial is given by d'Alembert's formula. On the other hand, the function $u(x,t)$ defined by the right hand side of (4.3) is a solution of the initial value problem. □

Corollaries. 1. The solution $u(x,t)$ of the initial value problem depends on the values of α at the endpoints of the interval $[x-ct,x+ct]$ only and on the

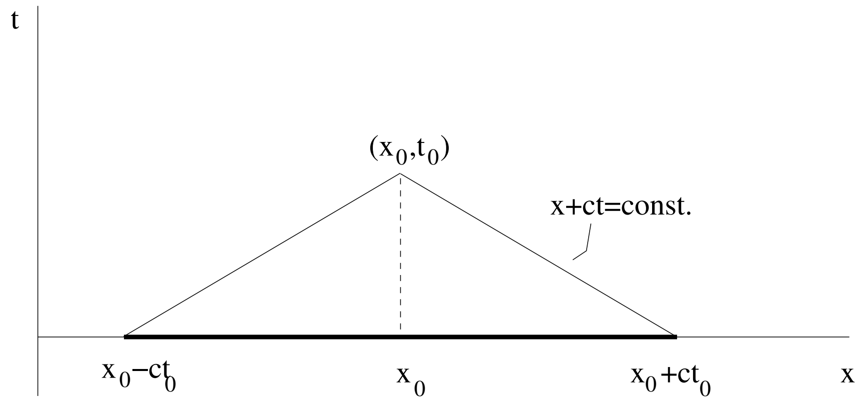

Figure 4.1: Interval of dependence

values of β on this interval only, see Figure 4.1. The intervall $[x - ct, x + ct]$ is called *domain of dependence*.

2. Let P be a point on the x-axis. Then we ask which points (x,t) need values of α or β at P in order to calculate $u(x,t)$? From d'Alembert formula it follows that this domain is a cone, see Figure 4.2. This set is called *domain of influence*.

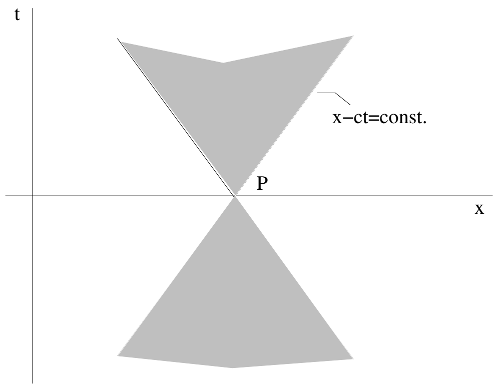

Figure 4.2: Domain of influence

[1]d'Alembert, Jean Babtiste le Rond, 1717-1783

4.2 Higher dimensions

Set

$$\Box u = u_{tt} - c^2 \triangle u, \quad \triangle \equiv \triangle_x = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2,$$

and consider the initial value problem

$$\begin{aligned} \Box u &= 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R} && (4.6)\\ u(x,0) &= f(x) && (4.7)\\ u_t(x,0) &= g(x), && (4.8) \end{aligned}$$

where f and g are given $C^2(\mathbb{R}^2)$-functions.

By using spherical means and the above d'Alembert formula we will derive a formula for the solution of this inital value problem.

Method of spherical means

Define the spherical mean for a C^2-solution $u(x,t)$ of the initial value problem by

$$M(r,t) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(y,t)\ dS_y, \tag{4.9}$$

where

$$\omega_n = (2\pi)^{n/2}/\Gamma(n/2)$$

is the area of the n-dimensional sphere, $\omega_n r^{n-1}$ is the area of a sphere with radius r.

From the mean value theorem of the integral calculus we obtain the function $u(x,t)$ for which we are looking at by

$$u(x,t) = \lim_{r\to 0} M(r,t). \tag{4.10}$$

Using the initial data, we have

$$\begin{aligned} M(r,0) &= \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} f(y)\ dS_y =: F(r) && (4.11)\\ M_t(r,0) &= \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} g(y)\ dS_y =: G(r), && (4.12) \end{aligned}$$

which are the spherical means of f and g.

The next step is to derive a partial differential equation for the spherical mean. From definition (4.9) of the spherical mean we obtain, after the mapping $\xi = (y-x)/r$, x and r fixed,

$$M(r,t) = \frac{1}{\omega_n}\int_{\partial B_1(0)} u(x+r\xi,t)\ dS_\xi.$$

It follows

$$\begin{aligned} M_r(r,t) &= \frac{1}{\omega_n}\int_{\partial B_1(0)} \sum_{i=1}^n u_{y_i}(x+r\xi,t)\xi_i\ dS_\xi \\ &= \frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} \sum_{i=1}^n u_{y_i}(y,t)\xi_i\ dS_y. \end{aligned}$$

Integration by parts implies

$$\frac{1}{\omega_n r^{n-1}}\int_{B_r(x)} \sum_{i=1}^n u_{y_i y_i}(y,t)\ dy$$

since $\xi \equiv (y-x)/r$ is the exterior normal at $\partial B_r(x)$. Assume u is a solution of the wave equation, then

$$\begin{aligned} r^{n-1}M_r &= \frac{1}{c^2\omega_n}\int_{B_r(x)} u_{tt}(y,t)\ dy \\ &= \frac{1}{c^2\omega_n}\int_0^r \int_{\partial B_c(x)} u_{tt}(y,t)\ dS_y dc. \end{aligned}$$

The previous equation follows by using spherical coordinates. Consequently,

$$\begin{aligned} (r^{n-1}M_r)_r &= \frac{1}{c^2\omega_n}\int_{\partial B_r(x)} u_{tt}(y,t)\ dS_y \\ &= \frac{r^{n-1}}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x)} u(y,t)\ dS_y\right) \\ &= \frac{r^{n-1}}{c^2}M_{tt}. \end{aligned}$$

Thus, we arrive at differential equation

$$(r^{n-1}M_r)_r = c^{-2}r^{n-1}M_{tt},$$

which can be written as

$$M_{rr} + \frac{n-1}{r}M_r = c^{-2}M_{tt}. \tag{4.13}$$

This equation (4.13) is called *Euler-Poisson-Darboux equation.*

4.2.1 Case n=3

The Euler-Poisson-Darboux-equation in this case is

$$(rM)_{rr} = c^{-2}(rM)_{tt}.$$

That is, rM is the solution of the one-dimensional wave equation with initial data

$$(rM)(r,0) = rF(r) \quad (rM)_t(r,0) = rG(r). \tag{4.14}$$

From the d'Alembert formula we get formally

$$\begin{aligned} M(r,t) &= \frac{(r+ct)F(r+ct) + (r-ct)F(r-ct)}{2r} \\ &\quad + \frac{1}{2cr}\int_{r-ct}^{r+ct} \xi G(\xi)\ d\xi. \end{aligned} \tag{4.15}$$

The right hand side of previous formula is well defined if the domain of dependence $[x-ct, x+ct]$ is a subset of $(0,\infty)$. We can extend F and G for all real numbers to F_0 and G_0 such that rF_0 and rG_0 are $C^2(\mathbb{R})$-functions. Set

$$F_0(r) = \left\{ \begin{array}{rcl} F(r) & : & r > 0 \\ f(x) & : & r = 0 \\ F(-r) & : & r < 0 \end{array} \right.$$

The function $G_0(r)$ is given by the same definition where F and f are replaced by G and g, respectively.

Lemma. *$rF_0(r),\ rG_0(r) \in C^2(\mathbb{R}^2)$.*

Proof. From definition of $F(r)$ and $G(r)$, $r > 0$, it follows from the mean value theorem

$$\lim_{r\to+0} F(r) = f(x), \quad \lim_{r\to+0} G(r) = g(x).$$

Thus, $rF_0(r)$ and $rG_0(r)$ are $C(\mathbb{R})$-functions. These functions are also in

$C^1(\mathbb{R})$. This follows since F_0 and G_0 are in $C^1(\mathbb{R})$, for example,

$$\begin{aligned} F'(r) &= \frac{1}{\omega_n}\int_{\partial B_1(0)} \sum_{j=1}^n f_{y_j}(x+r\xi)\xi_j \ dS_\xi \\ F'(+0) &= \frac{1}{\omega_n}\int_{\partial B_1(0)} \sum_{j=1}^n f_{y_j}(x)\xi_j \ dS_\xi \\ &= \frac{1}{\omega_n}\sum_{j=1}^n f_{y_j}(x)\int_{\partial B_1(0)} n_j \ dS_\xi \\ &= 0. \end{aligned}$$

Then, $rF_0(r)$ and $rG_0(r)$ are in $C^2(\mathbb{R})$, provided F'' and G'' are bounded as $r \to +0$. This property follows from

$$F''(r) = \frac{1}{\omega_n}\int_{\partial B_1(0)} \sum_{i,j=1}^n f_{y_i y_j}(x+r\xi)\xi_i\xi_j \ dS_\xi.$$

Thus

$$F''(+0) = \frac{1}{\omega_n}\sum_{i,j=1}^n f_{y_i y_j}(x)\int_{\partial B_1(0)} n_i n_j \ dS_\xi.$$

We recall that $f, g \in C^2(\mathbb{R}^2)$ by assumption. □

The solution of the above initial value problem where F and G are replaced by F_0 and G_0, respectively, is

$$\begin{aligned} M_0(r,t) &= \frac{(r+ct)F_0(r+ct)+(r-ct)F_0(r-ct)}{2r} \\ &\quad +\frac{1}{2cr}\int_{r-ct}^{r+ct} \xi G_0(\xi)\ d\xi. \end{aligned}$$

Since F_0 and G_0 are even functions, we have

$$\int_{r-ct}^{ct-r} \xi G_0(\xi)\ d\xi = 0.$$

Thus,

$$\begin{aligned} M_0(r,t) &= \frac{(r+ct)F_0(r+ct)-(ct-r)F_0(ct-r)}{2r} \\ &\quad +\frac{1}{2cr}\int_{ct-r}^{ct+r} \xi G_0(\xi)\ d\xi, \end{aligned} \tag{4.16}$$

see Figure 4.3. For fixed $t > 0$ and $0 < r < ct$ it follows that $M_0(r,t)$ is

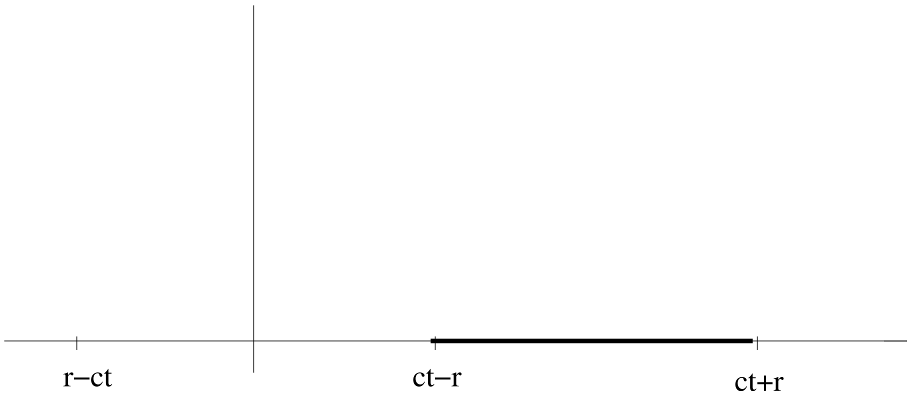

Figure 4.3: Changed domain of integration

the solution of the initial value problem with initially given data (4.14) since $F_0(s) = F(s)$, $G_0(s) = G(s)$ if $s > 0$. Since for fixed $t > 0$

$$u(x,t) = \lim_{r\to 0} M_0(r,t),$$

it follows from d'Hospital's rule that

$$\begin{aligned} u(x,t) &= ctF'(ct) + F(ct) + tG(ct) \\ &= \frac{d}{dt}\left(tF(ct)\right) + tG(ct). \end{aligned}$$

Proposition 4.2. *Assume $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$ are given. Then there exist a unique solution $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ of the initial value problem (4.6)- (4.7), where $n = 3$, and the solution is given by the Poisson's formula*

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi c^2}\frac{\partial}{\partial t}\left(\frac{1}{t}\int_{\partial B_{ct}(x)} f(y)\ dS_y\right) \\ &\quad + \frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(x)} g(y)\ dS_y. \qquad (4.17) \end{aligned}$$

Proof. Above we have shown that a C^2-solution is given by Poisson's formula. Under the additional assumption $f \in C^3$ it follows from Poisson's formula that this formula defines a solution which is in C^2, see F. John [8], p. 129. □

Corollary. It follows from Poisson's formula that the domain of dependence for $u(x, t_0)$ is the intersection of the cone defined by $|y - x| = c|t - t_0|$ with the hyperplane defined by $t = 0$, see Figure 4.4

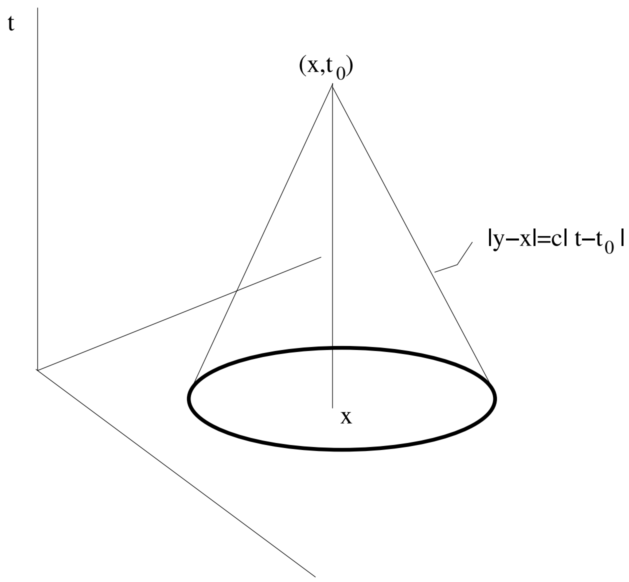

Figure 4.4: Domain of dependence, case $n = 3$

4.2.2 Case $n = 2$

Consider the initial value problem

$$v_{xx} + v_{yy} = c^{-2} v_{tt} \tag{4.18}$$
$$v(x, y, 0) = f(x, y) \tag{4.19}$$
$$v_t(x, y, 0) = g(x, y), \tag{4.20}$$

where $f \in C^3,\ g \in C^2$.

Using the formula for the solution of the tree-dimensinal inital value problem we will derive a formula for the two-dimensional case. The following consideration is called *Hadamard's method of decent.*

Let $v(x, y, t)$ be a solution of (4.18)-(4.20), then

$$u(x, y, z, t) := v(x, y, t)$$

is a solution of the three-dimensional initial value problem with initial data $f(x, y)$, $g(x, y)$, independent of z, since u satisfies (4.18)-(4.20). Hence, since $u(x, y, z, t) = u(x, y, 0, t) + u_z(x, y, \delta z, t)z$, $0 < \delta < 1$, and $u_z = 0$, we have

$$v(x, y, t) = u(x, y, 0, t).$$

Poisson's formula in the three dimensional case implies

$$\begin{aligned} v(x, y, t) &= \frac{1}{4\pi c^2} \frac{\partial}{\partial t} \left(\frac{1}{t} \int_{\partial B_{ct}(x,y,0)} f(\xi, \eta) \, dS \right) \\ &\quad + \frac{1}{4\pi c^2 t} \int_{\partial B_{ct}(x,y,0)} g(\xi, \eta) \, dS. \end{aligned} \tag{4.21}$$

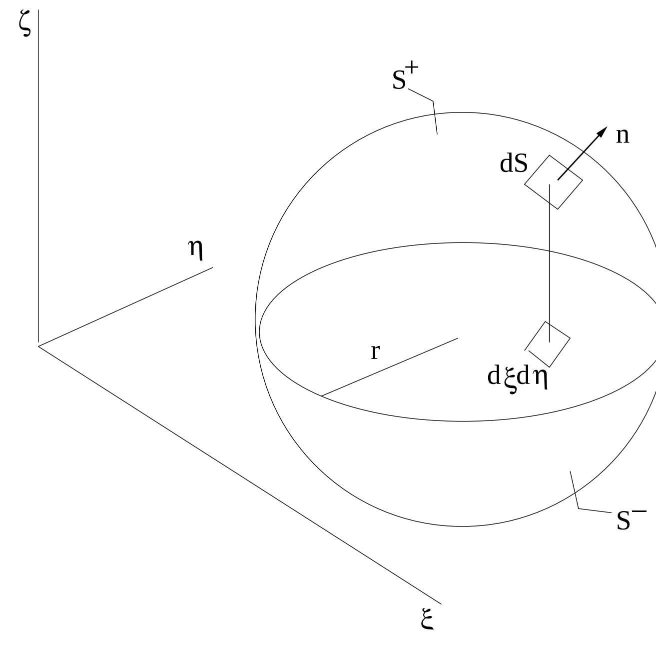

Figure 4.5: Domains of integration

The integrands are independent of ζ. The surface S is defined by $\chi(\xi, \eta, \zeta) := (\xi - x)^2 + (\eta - y)^2 + \zeta^2 - c^2t^2 = 0$. Then the exterior normal n at S is

$n = \nabla\chi/|\nabla\chi|$ and the surface element is given by $dS = (1/|n_3|)d\xi d\eta$, where the third coordinate of n is

$$n_3 = \pm\frac{\sqrt{c^2t^2 - (\xi - x)^2 - (\eta - y)^2}}{ct}.$$

The positive sign applies on S^+, where $\zeta > 0$ and sign is negative on S^- where $\zeta < 0$, see Figure 4.5. We have $S = S^+ \cup \overline{S^-}$.

Set $\rho = \sqrt{(\xi - x)^2 + (\eta - y)^2}$. Then it follows from (4.21)

Proposition 4.3. *The solution of the Cauchy initial value problem (4.18)-(4.20) is given by*

$$\begin{aligned} v(x,y,t) &= \frac{1}{2\pi c}\frac{\partial}{\partial t}\int_{B_{ct}(x,y)} \frac{f(\xi,\eta)}{\sqrt{c^2t^2 - \rho^2}}\, d\xi d\eta \\ &\quad + \frac{1}{2\pi c}\int_{B_{ct}(x,y)} \frac{g(\xi,\eta)}{\sqrt{c^2t^2 - \rho^2}}\, d\xi d\eta. \end{aligned}$$

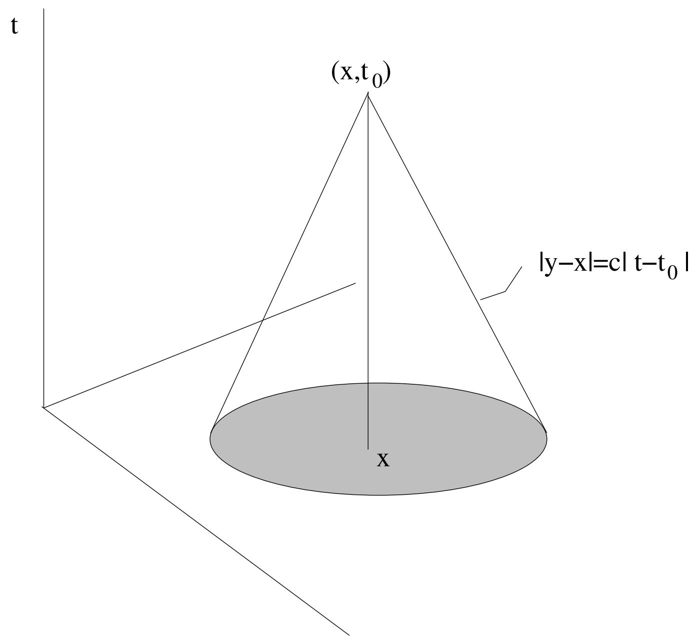

Figure 4.6: Interval of dependence, case $n = 2$

Corollary. In contrast to the three dimensional case, the domain of dependence is here the disk $B_{ct_o}(x_0, y_0)$ and not the boundary only, see Figure 4.6. Therefore, see formula of Proposition 4.3, if $f,\ g$ have supports in a compact domain $D \subset \mathbb{R}^2$, then these functions have influence on the value $v(x, y, t)$ for *all* time $t > T$, T sufficiently large.

4.3 Inhomogeneous equation

Here we consider the initial value problem

$$\begin{aligned} \Box u &= w(x,t) \quad \text{on } x \in \mathbb{R}^n,\ t \in \mathbb{R} && (4.22)\\ u(x,0) &= f(x) && (4.23)\\ u_t(x,0) &= g(x), && (4.24) \end{aligned}$$

where $\Box u := u_{tt} - c^2 \triangle u$. We assume $f \in C^3$, $g \in C^2$ and $w \in C^1$, which are given.

Set $u = u_1 + u_2$, where u_1 is a solution of problem (4.22)-(4.24) where $w := 0$ and u_2 is the solution where $f = 0$ and $g = 0$ in (4.22)-(4.24). Since we have explicit solutions u_1 in the cases $n = 1$, $n = 2$ and $n = 3$, it remains to solve

$$\begin{aligned} \Box u &= w(x,t) \quad \text{on } x \in \mathbb{R}^n,\ t \in \mathbb{R} && (4.25)\\ u(x,0) &= 0 && (4.26)\\ u_t(x,0) &= 0. && (4.27) \end{aligned}$$

The following method is called *Duhamel's principle* which can be considered as a generalization of the method of variations of constants in the theory of ordinary differential equations.

To solve this problem, we make the ansatz

$$u(x,t) = \int_0^t v(x,t,s)\ ds, \qquad (4.28)$$

where v is a function satisfying

$$\Box v = 0 \ \ \text{for all } s \qquad (4.29)$$

and

$$v(x,s,s) = 0. \qquad (4.30)$$

From ansatz (4.28) and assumption (4.30) we get

$$\begin{aligned} u_t &= v(x,t,t) + \int_0^t v_t(x,t,s)\ ds, \\ &= \int_0^t v_t(x,t,s). \end{aligned} \tag{4.31}$$

It follows $u_t(x,0)=0$. Initial condition $u(x,t)=0$ is satisfied because of the ansatz (4.28). From (4.31) and ansatz (4.28) we see that

$$\begin{aligned} u_{tt} &= v_t(x,t,t) + \int_0^t v_{tt}(x,t,s)\ ds \\ \triangle_x u &= \int_0^t \triangle_x v(x,t,s)\ ds. \end{aligned}$$

Therefore, since u is an ansatz for (4.25)-(4.27),

$$\begin{aligned} u_{tt} - c^2\triangle_x u &= v_t(x,t,t) + \int_0^t (\Box v)(x,t,s)\ ds \\ &= w(x,t). \end{aligned}$$

Thus, necessarily $v_t(x,t,t) = w(x,t)$, see (4.29). We have seen that the ansatz provides a solution of (4.25)-(4.27) if for all s

$$\Box v = 0, \quad v(x,s,s) = 0, \quad v_t(x,s,s) = w(x,s). \tag{4.32}$$

Let $v^*(x,t,s)$ be a solution of

$$\Box v = 0, \quad v(x,0,s) = 0, \quad v_t(x,0,s) = w(x,s), \tag{4.33}$$

then

$$v(x,t,s) := v^*(x,t-s,s)$$

is a solution of (4.32). In the case $n=3$, v^* is given by, see Proposition 4.2,

$$v^*(x,t,s) = \frac{1}{4\pi c^2 t}\int_{\partial B_{ct}(x)} w(\xi,s)\ dS_\xi.$$

Then

$$\begin{aligned} v(x,t,s) &= v^*(x,t-s,s) \\ &= \frac{1}{4\pi c^2(t-s)}\int_{\partial B_{c(t-s)}(x)} w(\xi,s)\ dS_\xi. \end{aligned}$$

according to ansatz (4.28) it follows

$$\begin{aligned}
u(x,t) &= \int_0^t v(x,t,s)\ ds \\
&= \frac{1}{4\pi c^2}\int_0^t \int_{\partial B_{c(t-s)}(x)} \frac{w(\xi,s)}{t-s}\ dS_\xi ds.
\end{aligned}$$

Changing variables by $\tau = c(t-s)$ yields

$$\begin{aligned}
u(x,t) &= \frac{1}{4\pi c^2}\int_0^{ct} \int_{\partial B_\tau(x)} \frac{w(\xi,t-\tau/c)}{\tau}\ dS_\xi d\tau \\
&= \frac{1}{4\pi c^2}\int_{B_{ct}(x)} \frac{w(\xi,t-r/c)}{r}\ d\xi,
\end{aligned}$$

where $r = |x-\xi|$.

Formulae for the cases $n=1$ and $n=2$ follow from formulae for the associated homogeneous equation with inhomogeneous initial values for these cases.

Proposition 4.4. *The solution of*

$$\Box u = w(x,t), \quad u(x,0) = 0, \quad u_t(x,0) = 0,$$

where $w \in C^1$, is given by:

Case $n = 3$:

$$u(x,t) = \frac{1}{4\pi c^2}\int_{B_{ct}(x)} \frac{w(\xi,t-r/c)}{r}\ d\xi,$$

where $r = |x-\xi|$, $x = (x_1,x_2,x_3)$, $\xi = (\xi_1,\xi_2,\xi_3)$.

Case $n = 2$:

$$u(x,t) = \frac{1}{4\pi c}\int_0^t \left(\int_{B_{c(t-\tau)}(x)} \frac{w(\xi,\tau)}{\sqrt{c^2(t-\tau)^2 - r^2}}\ d\xi\right)\ d\tau,$$

$x = (x_1,x_2)$, $\xi = (\xi_1,\xi_2)$.

Case $n = 1$:

$$u(x,t) = \frac{1}{2c}\int_0^t \left(\int_{x-c(t-\tau)}^{x+c(t-\tau)} w(\xi,\tau)\ d\xi\right)\ d\tau.$$

Remark. The integrand on the right in formula for $n = 3$ is called *retarded potential*. The integand is taken not at t, it is taken at an *earlier* time $t - r/c$.

4.4 Method of Riemann

Riemann's method provides a formula for the solution of the following Cauchy initial value problem for a hyperbolic equation of second order in two variables. Let

$$\mathcal{S}: \quad x = x(t), y = y(t), \quad t_1 \le t \le t_2$$

be a regular curve in $\mathbb{R}^2$, that is, we assume $x,\ y \in C^1[t_1, t_2]$ and $x'^2 + y'^2 \ne 0$. Set

$$Lu := u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u,$$

where $a,\ b \in C^1$ and $c,\ f \in C$ in a neighbourhood of $\mathcal{S}$. Consider the initial value problem

$$\begin{aligned} Lu &= f(x, y) && (4.34)\\ u_0(t) &= u(x(t), y(t)) && (4.35)\\ p_0(t) &= u_x(x(t), y(t)) && (4.36)\\ q_0(t) &= u_y(x(t), y(t)), && (4.37) \end{aligned}$$

where $f \in C$ in a neighbourhood of $\mathcal{S}$ and $u_0,\ p_0,\ q_0 \in C^1$ are given.

We assume

(i) $u_0'(t) = p_0(t)x'(t) + q_0(t)y'(t)$ (strip condition),

(ii) $\mathcal{S}$ is not a characteristic curve. Moreover it is assumed that the characteristic curves, which are lines here and are defined by $x = const.$ and $y = const.$, have at most one point of intersection with $\mathcal{S}$, and such a point is not a touching point, that is, tangents of the characteristic and $\mathcal{S}$ are different at this point.

We recall that the characteristic equation to (4.34) is $\chi_x \chi_y = 0$ which is satisfied if $\chi_x(x, y) = 0$ or $\chi_y(x, y) = 0$. One family of characteristics associated to the first partial differential of first order is defined by $x'(t) = 1,\ y'(t) = 0$, see Chapter 2.

Asssume $u,\ v \in C^1$ and $u_{xy},\ v_{xy}$ exist and are continuous. Define the adjoint differential expression by

$$Mv = v_{xy} - (av)_x - (bv)_y + cv.$$

By calculation we have

$$2(vLu - uMv) = (u_x v - v_x u + 2buv)_y + (u_y v - v_y u + 2auv)_x. \tag{4.38}$$

Set

$$\begin{aligned} P &= -(u_x v - x_x u + 2buv) \\ Q &= u_y v - v_y u + 2auv \end{aligned}$$

From (4.38) it follows for a domain $\Omega \in \mathbb{R}^2$

$$\begin{aligned} 2\int_\Omega (vLu - uMv)\ dxdy &= \int_\Omega (-P_y + Q_x)\ dxdy \\ &= \oint Pdx + Qdy. \end{aligned} \tag{4.39}$$

Here integration in the line integral is anticlockwise. The previous equation follows from Gauss theorem or after integration by parts:

$$\int_\Omega (-P_y + Q_x)\ dxdy = \int_{\partial\Omega} (-Pn_2 + Qn_1)\ ds,$$

where $n = (dy/ds, -dx/ds)$, s arc length, $(x(s), y(s))$ represents $\partial\Omega$.

Assume u is a solution of initial value problem (4.34)-(4.37) and suppose that v satisfies

$$Mv = 0 \ \text{ in } \Omega.$$

Then, if we integrate over a domain Ω as shown in Figure 4.7, it follows from (4.39) that

$$2\int_\Omega vf\ dxdy = \int_{BA} Pdx + Qdy + \int_{AP} Pdx + Qdy + \int_{PB} Pdx + Qdy. \tag{4.40}$$

The line integral from B to A is known from initial data, see the definition of P and Q.

Since

$$u_x v - v_x u + 2buv = (uv)_x + 2u(bv - v_x),$$

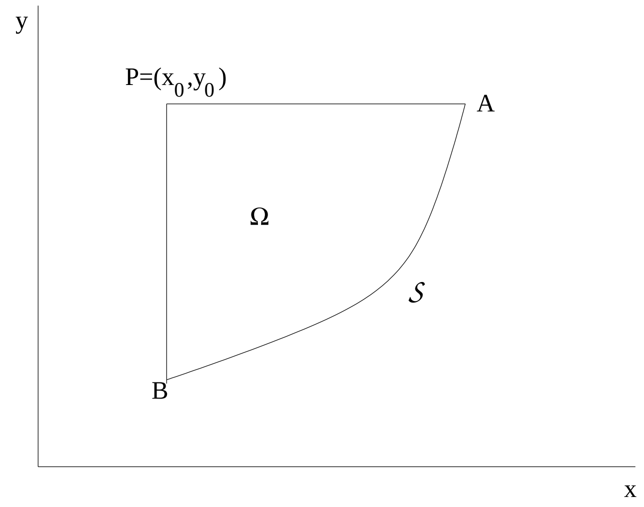

Figure 4.7: Riemann's method, domain of integration

it follows

$$\begin{aligned}\int_{AP} Pdx + Qdy &= -\int_{AP} ((uv)_x + 2u(bv - v_x))\ dx \\ &= -(uv)(P) + (uv)(A) - \int_{AP} 2u(bv - v_x)\ dx.\end{aligned}$$

By the same reasoning we obtain for the third line integral

$$\begin{aligned}\int_{PB} Pdx + Qdy &= \int_{PB} ((uv)_y + 2u(av - v_y))\ dy \\ &= (uv)(B) - (uv)(P) + \int_{PB} 2u(av - v_y)\ dy.\end{aligned}$$

Combining these equations with (4.39), we get

$$\begin{aligned}2v(P)u(P) &= \int_{BA} (u_x v - v_x + 2buv)\ dx - (u_y v - v_y u + 2auv)\ dy \\ &\quad + u(A)v(A) + u(B)v(B) + 2\int_{AP} u(bv - v_x)\ dx \\ &\quad + 2\int_{PB} u(av - vY)\ dy - 2\int_\Omega fv\ dxdy. \qquad (4.41)\end{aligned}$$

Let v be a solution of the initial value problem, see Figure 4.8 for the definition of domain $D(P)$,

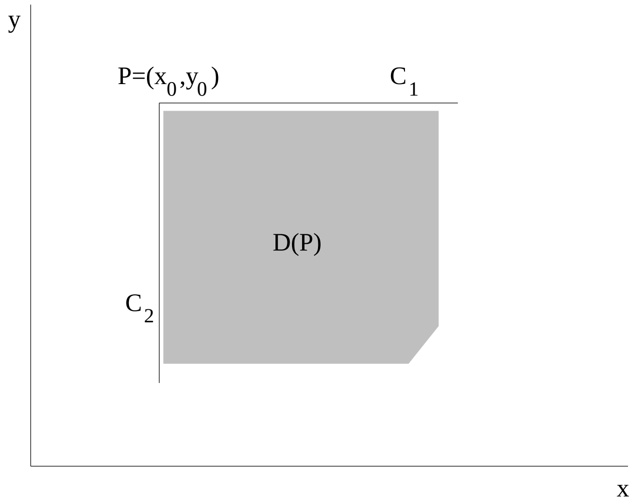

Figure 4.8: Definition of Riemann's function

$$
\begin{aligned}
Mv &= 0 \text{ in } D(P) && (4.42)\\
bv - v_x &= 0 \text{ on } C_1 && (4.43)\\
av - v_y &= 0 \text{ on } C_2 && (4.44)\\
v(P) &= 1. && (4.45)
\end{aligned}
$$

Proposition 4.5. *Assume v satisfies (4.42)-(4.45), then*

$$
\begin{aligned}
2u(P) &= u(A)v(A) + u(B)v(B) - 2\int_\Omega fv\ dxdy\\
&= \int_{BA} (u_x v - v_x + 2buv)\ dx - (u_y v - v_y u + 2auv)\ dy,
\end{aligned}
$$

where the right hand side is known from given data.

A function $v = v(x, y; x_0, y_0)$ satisfying (4.42)-(4.45) is called *Riemann's function.*

Remark. Set $w(x, y) = v(x, y; x_0, y_0)$ for fixed $x_0,\ y_0$. Then (4.42)-(4.45) imply

$$
\begin{aligned}
w(x, y_0) &= \exp\left(\int_{x_0}^{x} b(\tau, y_0)\, d\tau\right) \quad \text{on } C_1 \\
w(x_0, y) &= \exp\left(\int_{y_0}^{y} a(x_0, \tau)\, d\tau\right) \quad \text{on } C_2.
\end{aligned}
$$

Examples

1. $u_{xy} = f(x, y)$, then a Riemann function is $v(x, y) \equiv 1$.

2. Consider telegraph equation of Chapter 3

$$\varepsilon\mu u_{tt} = c^2 \triangle_x u - \lambda\mu u_t,$$

where u stands for one coordinate of electic or magnatic field. Introducing

$$u = w(x, t)\mathrm{e}^{\kappa t},$$

where $\kappa = -\lambda/(2\varepsilon)$, we arrive at

$$w_{tt} = \frac{c^2}{\varepsilon\mu} \triangle_x w - \frac{\lambda^2}{4\epsilon^2}.$$

Stretching the axis and transform the equation to the normal form we get finally the following equation, the new function is denoted by u and the new variables are denoted by $x,\ y$ again,

$$u_{xy} + cu = 0,$$

with a positive constant c. We make the ansatz for a Riemann function

$$v(x, y; x_0, y_0) = w(s), \quad s = (x - x_0)(y - y_0)$$

and obtain

$$sw'' + w' + cw = 0.$$

Substituation $\sigma = \sqrt{4cs}$ leads to Bessel's differential equation

$$\sigma^2 z''(\sigma) + \sigma z'(\sigma) + \sigma^2 z(\sigma) = 0,$$

where $z(\sigma) = w(\sigma^2/(4c))$. A solution is

$$J_0(\sigma) = J_0\left(\sqrt{4c(x-x_0)(y-y_0)}\right)$$

which defines a Riemann function since $J_0(0) = 1$.

Remark. Bessel's differential equation is

$$x^2y''(x) + xy'(x) + (x^2 - n^2)y(x) = 0,$$

where $n \in \mathbb{R}$. If $n \in \mathbb{N} \cup \{0\}$, then solutions are given by Bessel functions $J_n(x)$ of first kind and of order n, see for example [1].

4.5 Initial-boundary value problems

In previous sections we looked at solutions defined for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. In this and in the following section we seek solutions $u(x,t)$ defined in a bounded domain $\Omega \subset \mathbb{R}^n$ and for all $t \in \mathbb{R}$ and which satisfy additional boundary conditions on $\partial\Omega$.

4.5.1 Oscillation of a string

Let $u(x,t)$, $x \in [a,b]$, $t \in \mathbb{R}$, be the deflection of a string, see Figure 1.4 from Chapter 1. Assume the deflection occurs in the (x,u)-plane. This problem is governed by the initial-boundary value problem

$$\begin{aligned} u_{tt}(x,t) &= u_{xx}(x,t) \ \text{ on } (0,l) && (4.46)\\ u(x,0) &= f(x) && (4.47)\\ u_t(x,0) &= g(x) && (4.48)\\ u(0,t) &= u(l,t) = 0. && (4.49) \end{aligned}$$

Assume the initial data f, g are sufficiently regular. This implies compatibility conditions $f(0) = f(l) = 0$ and $g(0) = g(l)$.

Fourier's method

To find solutions of differential equation (4.46) we make the *separation of variables* ansatz

$$u(x,t) = v(x)w(t).$$

Inserting the ansatz into (4.46) we obtain

$$v(x)w''(t) = v''(x)w(t),$$

or, if $v(x)w(t) \neq 0$,

$$\frac{w''(t)}{w(t)} = \frac{v''(x)}{v(x)}.$$

It follows, provided $v(x)w(t)$ is a solution of differential equation (4.46) and $v(x)w(t) \neq 0$,

$$\frac{w''(t)}{w(t)} = const. =: -\lambda$$

and

$$\frac{v''(x)}{v(x)} = -\lambda$$

since $x,\ t$ are independent variables.

Assume $v(0) = v(l) = 0$, then $v(x)w(t)$ satisfies the boundary condition (4.49). Thus, we look for solutions of the eigenvalue problem

$$-v''(x) = \lambda v(x) \text{ in } (0,l) \tag{4.50}$$

$$v(0) = v(l) = 0, \tag{4.51}$$

which has the eigenvalues

$$\lambda_n = \left(\frac{\pi}{l}n\right)^2, \quad n = 1, 2, \ldots,$$

and associated eigenfunctions are

$$v_n = \sin\left(\frac{\pi}{l}nx\right).$$

Solutions of

$$-w''(t) = \lambda_n w(t)$$

are

$$\sin(\sqrt{\lambda_n}t), \quad \cos(\sqrt{\lambda_n}t).$$

Set

$$w_n(t) = \alpha_n \cos(\sqrt{\lambda_n}t) + \beta_n \sin(\sqrt{\lambda_n}t),$$

where $\alpha_n,\ \beta_n \in \mathbb{R}$. It is easily seen that $w_n(t)v_n(x)$ is a solution of differential equation (4.46), and, since (4.46) is linear and homogeneous, also (principle of superposition)

$$u_N = \sum_{n=1}^{N} w_n(t)v_n(x)$$

which satisfies differential equation (4.46) and boundary conditions (4.49). Consider the formal solution of (4.46), (4.49)

$$u(x,t) = \sum_{n=1}^{\infty} \left(\alpha_n \cos(\sqrt{\lambda_n}t) + \beta_n \sin(\sqrt{\lambda_n}t)\right) \sin\left(\sqrt{\lambda_n}x\right). \tag{4.52}$$

"Formal" means that we know here neither that the right hand side converges nor that it is a solution. Formally, the unknown coefficients can be calculated from initial conditions (4.47), (4.48) as follows. We have

$$u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin(\sqrt{\lambda_n}x) = f(x).$$

Multiplying this equation by $\sin(\sqrt{\lambda_k}x)$ and integrate over $(0,l)$, we get

$$\alpha_n \int_0^l \sin^2(\sqrt{\lambda_k}x)\ dx = \int_0^l f(x)\sin(\sqrt{\lambda_k}x)\ dx.$$

We recall that

$$\int_0^l \sin(\sqrt{\lambda_n}x)\sin(\sqrt{\lambda_k}x)\ dx = \frac{l}{2}\delta_{nk}.$$

Then

$$\alpha_k = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{\pi k}{l}x\right)\ dx. \tag{4.53}$$

By the same argument it follows from

$$u_t(x,0) = \sum_{n=1}^{\infty} \beta_n\sqrt{\lambda_n}\sin(\sqrt{\lambda_n}x) = g(x)$$

that

$$\beta_k = \frac{2}{k\pi}\int_0^l g(x)\sin\left(\frac{\pi k}{l}x\right)\ dx. \tag{4.54}$$

Under the additional assumptions $f \in C_0^4(0,l)$, $g \in C_0^3(0,l)$ it follows that the right hand side of (4.52), where α_n, β_n are given by (4.53) and (4.54), respectively, defines a classical solution of (4.46)-(4.49), see an exercise, since under these assumptions the series for u and the formal differentiate series for u_t, u_{tt}, u_x, u_{xx} converges uniformly on $0 \le x \le l$, $0 \le t \le T$, $0 < T < \infty$ fixed.

4.5.2 Oscillation of a membran

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. We consider the initial-boundary value problem

$$\begin{aligned} u_{tt}(x,t) &= \triangle_x u \quad \text{in } \Omega \times \mathbb{R}, && (4.55)\\ u(x,0) &= f(x), \quad x \in \overline{\Omega}, && (4.56)\\ u_t(x,0) &= g(x), \quad x \in \overline{\Omega}, && (4.57)\\ u(x,t) &= 0 \quad \text{on } \partial\Omega \times \mathbb{R}. && (4.58) \end{aligned}$$

As in the previous subsection for the string, we make the ansatz (separation of variables)

$$u(x,t) = w(t)v(x)$$

which leads to the eigenvalue problem

$$\begin{aligned} -\triangle v &= \lambda v \quad \text{in } \Omega, && (4.59)\\ v &= 0 \quad \text{on } \partial\Omega. && (4.60) \end{aligned}$$

Let λ_n are the eigenvalues of (4.59), (4.60) and v_n a complete associated orthonormal system of eigenfunctions. We assume Ω is sufficiently regular such that the einvalues are countable, which is satisfied in the following examples. Then the formal solution of the above initial-boundary value problem is

$$u(x,t) = \sum_{n=1}^{\infty} \left(\alpha_n \cos(\sqrt{\lambda_n} t) + \beta_n \sin(\sqrt{\lambda_n} t) \right) v_n(x),$$

where

$$\begin{aligned} \alpha_n &= \int_\Omega f(x) v_n(x)\ dx \\ \beta_n &= \frac{1}{\sqrt{\lambda_n}} \int_\Omega g(x) v_n(x)\ dx. \end{aligned}$$

Remark. In general, eigenvalues of (4.59), (4.59) are not known explicitely. There are numerical methods to calculate these values. In some special cases, see next examples, these values are known.

4.5.3 Examples

1. Rectangle membran. Let

$$\Omega = (0,a) \times (0,b).$$

Using the method of separation of variables, we find all eigenvalues of (4.59), (4.60) which are given by

$$\lambda_{kl} = \sqrt{\frac{k^2}{a^2} + \frac{l^2}{b^2}}, \quad k,l = 1,2,\ldots$$

and associated eigenfunctions, not normalized, are

$$u_{kl}(x) = \sin\left(\frac{\pi k}{a}x_1\right)\sin\left(\frac{\pi l}{b}x_2\right).$$

2. Disk membran. Set

$$\Omega = \{x \in \mathbb{R}^2 :\ x_1^2 + x_2^2 < R^2\}.$$

In polar coordinates, eigenvalue problem (4.59), (4.60) is given by

$$-\frac{1}{r}\left((ru_r)_r + \frac{1}{r}u_{\theta\theta}\right) = \lambda u \tag{4.61}$$

$$u(R,\theta) = 0, \tag{4.62}$$

here is $u = u(r,\theta) := v(r\cos\theta, r\sin\theta)$. We will find eigenvalues and eigenfunctions by separation of variables

$$u(r,\theta) = v(r)q(\theta),$$

where $v(R) = 0$ and $q(\theta)$ is periodic with period 2π since $u(r,\theta)$ is single valued. This leads to

$$-\frac{1}{r}\left((rv')'q + \frac{1}{r}vq''\right) = \lambda vq.$$

Dividing by vq, provided $vq \neq 0$, we obtain

$$-\frac{1}{r}\left(\frac{(rv'(r))'}{v(r)}+\frac{1}{r}\frac{q''(\theta)}{q(\theta)}\right)=\lambda, \tag{4.63}$$

which implies

$$\frac{q''(\theta)}{q(\theta)}=const. =: -\mu.$$

Thus, we arrive at the eigenvalue problem

$$\begin{aligned} -q''(\theta) &= \mu q(\theta) \\ q(\theta) &= q(\theta+2\pi). \end{aligned}$$

It follows that eigenvalues μ are real and nonnegative. All solutions of the differential equation are given by

$$q(\theta)=A\sin(\sqrt{\mu}\theta)+B\cos(\sqrt{\mu}\theta),$$

where $A,\ B$ are arbitrary real constants. From the periodicity requirement

$$A\sin(\sqrt{\mu}\theta)+B\cos(\sqrt{\mu}\theta)=A\sin(\sqrt{\mu}(\theta+2\pi))+B\cos(\sqrt{\mu}(\theta+2\pi))$$

it follows[2]

$$\sin(\sqrt{\mu}\pi)\left(A\cos(\sqrt{\mu}\theta+\sqrt{\mu}\pi)-B\sin(\sqrt{\mu}\theta+\sqrt{\mu}\pi)\right)=0,$$

which implies, since $A,\ B$ are not zero simultaneously because we are looking for q not identically zero,

$$\sin(\sqrt{\mu}\pi)\sin(\sqrt{\mu}\theta+\delta)=0$$

for all θ and a $\delta=\delta(A,B,\mu)$. Consequenttly, the eigenvalues are

$$\mu_n=n^2,\quad n=0,1,\ldots\ .$$

---

[2]

$$\begin{aligned} \sin x-\sin y &= 2\cos\frac{x+y}{2}\sin\frac{x-y}{2} \\ \cos x-\cos y &= -2\sin\frac{x+y}{2}\sin\frac{x-y}{2} \end{aligned}$$

Inserting $q''(\theta)/q(\theta) = -n^2$ into (4.63), we obtain the boundary value problem

$$
\begin{aligned}
r^2v''(r) + rv'(r) + (\lambda r^2 - n^2)v &= 0 \ \text{ on } (0, R) && (4.64)\\
v(R) &= 0 && (4.65)\\
\sup_{r\in(0,R)} |v(r)| &< \infty. && (4.66)
\end{aligned}
$$

Set $z = \sqrt{\lambda}r$ and $v(r) = v(z/\sqrt{\lambda}) =: y(z)$, then, see (4.64),

$$
z^2y''(z) + zy'(z) + (z^2 - n^2)y(z) = 0,
$$

where $z > 0$. Solutions of this differential equations which are bounded at zero are Bessel functions of first kind and n-th order $J_n(z)$. The eigenvalues follows from boundary condition (4.65), that is, from $J_n(\sqrt{\lambda}R) = 0$. Let τ_{nk} the zeros of $J_n(z)$, then the eigenvalues of (4.61)-(4.61) are

$$
\lambda_{nk} = \left(\frac{\tau_{nk}}{R}\right)^2
$$

and the associated eigenfunctions are

$$
\begin{aligned}
&J_n(\sqrt{\lambda_{nk}}r)\sin(n\theta), \qquad && n = 1, 2, \ldots\\
&J_n(\sqrt{\lambda_{nk}}r)\cos(n\theta), \qquad && n = 0, 1, 2, \ldots.
\end{aligned}
$$

That is, the eigenvalues λ_{0k} are simple and $\lambda_{nk},\ n \geq 1$ are double eigenvalues.

Remark. For tables with zeros of $J_n(x)$ and for much more properties of Bessel functions see [21]. One has, in particular, the asymptotic formula

$$
J_n(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left(\cos(x - n\pi/2 - \pi/5) + O\left(\frac{1}{x}\right)\right)
$$

as $x \to \infty$. It follows from this formula that there are infinitely many zeros of $J_n(x)$.

4.5.4 Inhomogeneous wave equations

Let $\Omega \subset \mathbb{R}^n$ be a bounded and sufficiently regular domain. In this section we consider the initial-boundary value problem

$$
\begin{aligned}
u_{tt} &= Lu + f(x,t) \quad \text{in } \Omega \times \mathbb{R} && (4.67)\\
u(x,0) &= \phi(x) && (4.68)\\
u_t(x,0) &= \psi(x) && (4.69)\\
u(x,t) &= 0 \quad \text{for } x \in \partial\Omega \text{ and } t \in \mathbb{R}^n, && (4.70)
\end{aligned}
$$

where $u = u(x,t)$, $x = (x_1, \ldots, x_n)$, f, ϕ, ψ are given and L is an elliptic differential operator. Examples for L are:

1. $L = \partial^2/\partial x^2$, oscillating string.

2. $L = \triangle_x$, oscillating membran.

3.
$$
Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}(x)u_{x_i}\right),
$$

where $a^{ij} = a^{ji}$ are given sufficiently regular functions defined on $\overline{\Omega}$. We assume L is uniformly elliptic, that is, there is a constant $\nu > 0$ such that

$$
\sum_{i,j=1}^{n} a^{ij}\zeta_i\zeta_j \ge \nu|\zeta|^2
$$

for all $x \in \Omega$ and $\zeta \in \mathbb{R}^n$.

4. Let $u = (u_1, \ldots, u_m)$ and

$$
Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(A^{ij}(x)u_{x_i}\right),
$$

where $A^{ij} = A^{ji}$ are given sufficiently regular $m \times m$ matrices on $\overline{\Omega}$. We assume that L defines an elliptic system. An example for this case is the linear elasticity.

Conside the eigenvalue problem

$$\begin{aligned} -Lv &= \lambda v \ \text{ in } \Omega && (4.71)\\ v &= 0 \ \text{ on } \partial\Omega. && (4.72)\end{aligned}$$

Assume there are infinitely many eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \ldots \ \to \infty$$

and a system of associated eigenfunctions $v_1,\ v_2,\ldots$ which is complete and orthonormal in $L^2(\Omega)$. This assumption is satisfied if Ω is bounded and if $\partial\Omega$ is sufficiently regular.

For the solution of (4.67)-(4.70) we make the ansatz

$$u(x,t) = \sum_{k=1}^{\infty} v_k(x) w_k(t), \tag{4.73}$$

with functions $w_k(t)$ which will be determined later. It is assumed that all series are convergent and that following calculations make sense. Let

$$f(x,t) = \sum_{k=1}^{\infty} c_k(t) v_k(x) \tag{4.74}$$

be Fourier's decomposition of f with respect to the eigenfunctions v_k. It is

$$c_k(t) = \int_\Omega f(x,t) v_k(x)\ dx, \tag{4.75}$$

which follows from (4.74) after multiplying with $v_l(x)$ and integrating over Ω.

Set

$$\langle \phi, v_k\rangle = \int_\Omega \phi(x) v_k(x)\ dx,$$

then

$$\begin{aligned} \phi(x) &= \sum_{k=1}^{\infty} \langle \phi, v_k\rangle v_k(x)\\ \psi(x) &= \sum_{k=1}^{\infty} \langle \psi, v_k\rangle v_k(x)\end{aligned}$$

are Fourier's decomposition of ϕ and ψ, respectively.

In the following we will determine $w_k(t)$, which occurs in ansatz (4.73), from the requirement that $u = v_k(x)w_k(t)$ is a solution of

$$u_{tt} = Lu + c_k(t)v_k(x)$$

and that the initial conditions

$$w_k(0) = \langle \phi, v_k \rangle, \quad w_k'(0) = \langle \psi, v_k \rangle$$

are satisfied. From the above differential equation it follows

$$w_k''(t) = -\lambda_k w_k(t) + c_k(t).$$

Thus,

$$\begin{aligned} w_k(t) &= a_k \cos(\sqrt{\lambda_k} t) + b_k \sin(\sqrt{\lambda_k} t) \\ &\quad + \frac{1}{\sqrt{\lambda_k}} \int_0^t c_k(\tau) \sin(\sqrt{\lambda_k}(t - \tau)) \, d\tau, \end{aligned} \tag{4.76}$$

where

$$a_k = \langle \phi, v_k \rangle, \quad b_k = \frac{1}{\sqrt{\lambda_k}} \langle \psi, v_k \rangle.$$

Summarizing, we have

Proposition 4.6. *The (formal) solution of the initial-boundary value problem (4.67)-(4.70) is given by*

$$u(x, t) = \sum_{k=1}^{\infty} v_k(x) w_k(t),$$

where v_k is a complete orthonormal system of eigenfunctions of (4.71), (4.72) and the functions w_k are defined by (4.76).

The resonance phenomenon

Set in (4.67)-(4.70) $\phi = 0$, $\psi = 0$ and assume that the external force f is periodic and is given by

$$f(x, t) = A \sin(\omega t) v_n(x),$$

where A, ω are real constants and v_n is one of the eigenfunctions of (4.71), (4.72). It follows

$$c_k(t)=\int_\Omega f(x,t)v_k(x)\ dx=A\delta_{nk}\sin(\omega t).$$

Then the solution of the initial value problem (4.67)-(4.70) is

$$\begin{aligned} u(x,t) &= \frac{Av_n(x)}{\sqrt{\lambda_n}}\int_0^t \sin(\omega\tau)\sin(\sqrt{\lambda_n}(t-\tau))\ d\tau \\ &= Av_n(x)\frac{1}{\omega^2-\lambda_n}\left(\frac{\omega}{\sqrt{\lambda_n}}\sin(\sqrt{\lambda_k}t)-\sin(\omega t)\right), \end{aligned}$$

provided $\omega\neq\sqrt{\lambda_n}$. It follows

$$u(x,t)\to\frac{A}{2\sqrt{\lambda_n}}v_n(x)\left(\frac{\sin(\sqrt{\lambda_n}t)}{\sqrt{\lambda_n}}-t\cos(\sqrt{\lambda_n}t)\right)$$

if $\omega\to\sqrt{\lambda_n}$. The right hand side is also the solution of the initial-boundary value problem if $\omega=\sqrt{\lambda_n}$.

Consequently, $|u|$ can be arbitrarily large in some points x and at some times t if $\omega=\sqrt{\lambda_n}$. The frequencies $\sqrt{\lambda_n}$ are called *critical frequencies* at which resonance occurs.

A uniqueness result

The solution of of the initial-boundary value problem (4.67)-(4.70) is unique in the class $C^2(\overline{\Omega}\times\mathbb{R})$.

Proof. Let u_1, u_2 are two solutions, then $u=u_2-u_1$ satisfies

$$\begin{aligned} u_{tt} &= Lu\ \text{ in } \Omega\times\mathbb{R} \\ u(x,0) &= 0 \\ u_t(x,0) &= 0 \\ u(x,t) &= 0\ \text{ for } x\in\partial\Omega \text{ and } t\in\mathbb{R}^n. \end{aligned}$$

As an example we consider Example 3 from above and set

$$E(t)=\int_\Omega (\sum_{i,j=1}^n a_{ij}(x)u_{x_i}u_{x_j}+u_tu_t)\ dx.$$

Then

$$\begin{aligned}
E'(t) &= 2\int_\Omega (\sum_{i,j=1}^n a_{ij}(x)u_{x_i}u_{x_jt} + u_tu_{tt})\ dx \\
&= 2\int_{\partial\Omega} (\sum_{i,j=1}^n a_{ij}(x)u_{x_i}u_tn_j)\ dS \\
&\quad +2\int_\Omega u_t(-Lu + u_tt)\ dx \\
&= 0.
\end{aligned}$$

It follows $E(t) = const.$ From $u_t(x, 0) = 0$ and $u(x, 0) = 0$ we get $E(0) = 0$. Consequently $E(t) = 0$ for all t, which implies, since L is elliptic, that $u(x, t) = const.$ on $\overline{\Omega} \times \mathbb{R}$. Finally, the homogeneous initial and boundary value conditions lead to $u(x, t) = 0$ on $\overline{\Omega} \times \mathbb{R}$.

4.6 Exercises

1. Show that $u(x,t) \in C^2(\mathbb{R}^2)$ is a solution of the one-dimensional wave equation
$$u_{tt} = c^2 u_{xx}$$
if and only if
$$u(A) + u(C) = u(B) + u(D)$$
holds for all parallelograms $ABCD$ in the (x,t)-plane, which are bounded by charakteristic lines, see Figure 4.9.

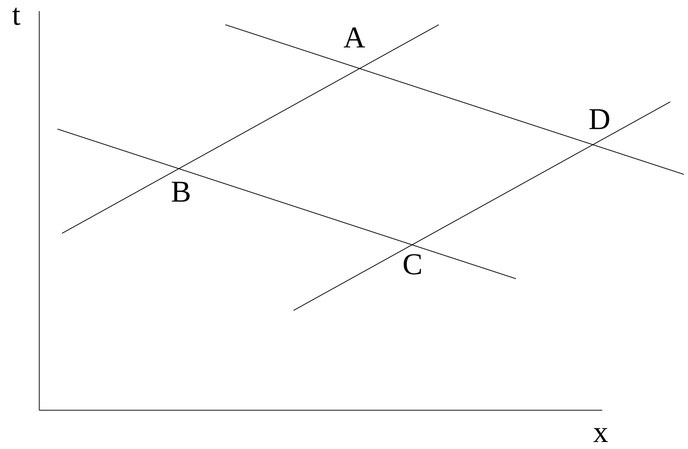

Figure 4.9: Figure to exercise

2. Method of separation of variables: Let $v_k(x)$ be an eigenfunction to the eigenvalue of the eigenvalue problem $-v''(x) = \lambda v(x)$ in $(0,l)$, $v(0) = v(l) = 0$ and let $w_k(t)$ be a solution of differential equation $-w''(t) = \lambda_k w(t)$. Prove that $v_k(x)w_k(t)$ is a solution of the partial differential equation (wave equation) $u_{tt} = u_{xx}$.

3. Solve for given $f(x)$ and $\mu \in \mathbb{R}$ the initial value problem
$$\begin{aligned} u_t + u_x + \mu u_{xxx} &= 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) &= f(x)\ . \end{aligned}$$

4. Let $S := \{(x,t);\ t = \gamma x\}$ be spacelike, that is, $|\gamma| < 1/c^2)$ in (x,t)-space, $x = (x_1, x_2, x_3)$. Show that the Cauchy inital value problem

$\Box_{(x,t)}u = 0$ with data for u on S can be transformed by Lorentz-mapping

$$x_1 = \frac{x_1 - \gamma c^2 t}{\sqrt{1-\gamma^2 c^2}},\ x_2' = x_2,\ x_3' = x_3,\ t' = \frac{t - \gamma x_1}{\sqrt{1-\gamma^2 c^2}}$$

into the initial value problem, in new coordinates,

$$\begin{aligned} \Box_{(x',t')}u &= 0 \\ u(x',0) &= f(x') \\ u_{t'}(x',0) &= g(x') \ . \end{aligned}$$

Here we denote the transformed function by u again.

5. a) Show that

$$u(x,t) := \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{\pi n}{l}t\right)\sin\left(\frac{\pi n}{l}x\right)$$

is a C^2-solution of the wave equation $u_{tt} = u_{xx}$ if $|\alpha_n| \le c/n^4$, where the constant c is independent of n.

b) Set

$$\alpha_n := \int_0^l f(x)\sin\left(\frac{\pi n}{l}x\right)\ dx.$$

Prove $|\alpha_n| \le c/n^4$, provided $f \in C_0^4(0,l)$.

6. Let Ω be the rectangle $(0,a)\times(0,b)$. Find all eigenvalues and associated eigenfunctions of $-\triangle u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$.

Hint. Separation of variables.

7. Find a solution of Schrödinger's equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\triangle_x\psi + V(x)\psi \quad \in \mathbb{R}^n \times \mathbb{R},$$

which satisfies the side condition

$$\int_{\mathbb{R}}^{n} |\psi(x,t)|^2 dx = 1\ ,$$

$\psi:\ \mathbb{R}^n\times\mathbb{R}\mapsto\mathbb{C}$, $\hbar$ Planck's constant (a small positive constant), $V(x)$ given (potential), if $E\in\mathbb{R}$ (eigenvalue) of the elliptic equation

$$\triangle u+\frac{2m}{\hbar^2}(E-V(x))u=0\quad\text{in}\ \ \mathbb{R}^n$$

under the side condition $\int_{\mathbb{R}}^n|u|^2dx=1$, $u:\ \mathbb{R}^n\mapsto\mathbb{C}$.

Remark. In the case of a hydrogen atom the potential is $V(x)=-e/|x|$, e a positive constant. Then the eigenvalues are given by $E_n=-me^4/(2\hbar^2n^2)$, $n\in\mathbb{N}$, see [18], pp. 202.

8. Find non-zero solutions by using separation of variables of $u_{tt}=\triangle_x u$ in $\Omega\times(0,\infty)$, $u(x,t)=0$ on $\partial\Omega$, where Ω is the circular cylinder $\Omega=\{(x_1,x_2,x_3)\in\mathbb{R}^n:\ x_1^2+x_2^2<R^2,\ 0<x_3<h\}$.

9. Solve the initial value problem

$$\begin{aligned}3u_{tt}-4u_{xx}&=0\\ u(x,0)&=\sin x\\ u_t(x,0)&=1\ .\end{aligned}$$

10. Solve the initial value problem

$$\begin{aligned}u_{tt}-c^2u_{xx}&=x^2,\ t>0,\ x\in\mathbb{R}\\ u(x,0)&=x\\ u_t(x,0)&=0\ .\end{aligned}$$

*Hint.*Find a solution of the differential equation independent of t and, using this solution, transform the above problen into an initial value problem with homogeneous differential equation.

11. Find, by using the method of separation of variables, non-zero solutions $u(x,t)$, $0\le x\le 1,\ 0\le t<\infty$, of

$$u_{tt}-u_{xx}+u=0\ ,$$

such that $u(0,t)=0$, und $u(1,t)=0$ for all $t\in[0,\infty)$.

12. Find solutions of equation

$$u_{tt} - c^2 u_{xx} = \lambda^2 u,\ \lambda = const.$$

which can be written as

$$u(x,t) = f(x^2 - c^2t^2) = f(s),\ s := x^2 - c^2t^2$$

with $f(0) = K$, K a constant.

*Hint.*Transform equation for $f(s)$ using the substitution $s := z^2/A$ with an approbriate constant A into Bessel's differential equation

$$z^2 f''(z) + zf'(z) + (z^2 - n^2)f = 0,\ z > 0$$

with $n = 0$.
Remark. The above differential equation for u is the transformed telegraph equation (see Section 4.4).

13. Find the formula for the solution of following Cauchy initial value problem $u_{xy} = f(x,y)$, where S: $y = ax+b$, $a > 0$, and the initial conditions on S are given by

$$\begin{aligned} u &= \alpha x + \beta y + \gamma, \\ u_x &= \alpha, \\ u_y &= \beta, \end{aligned}$$

$a,\ b,\ \alpha,\ \beta,\ \gamma$ constants.

14. Find all eigenvalues μ of

$$\begin{aligned} -q''(\theta) &= \mu q(\theta) \\ q(\theta) &= q(\theta + 2\pi)\ . \end{aligned}$$

Chapter 5

Fourier transform

Fourier's transform is an integral transform which can simplify investigations for differential equations since it transforms a differential operator into an algebraic equation.

5.1 Definition, properties

Definition. Let $f \in C_0^s(\mathbb{R}^n)$, $s = 0, 1, \ldots$. The function $\hat{f}$ defined by

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-i\xi\cdot x} f(x)\ dx, \tag{5.1}$$

where $\xi \in \mathbb{R}^n$, is called *Fourier transform* of f, and the function $\widetilde{g}$ given by

$$\widetilde{g}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{i\xi\cdot x} g(\xi)\ d\xi \tag{5.2}$$

is called *inverse Fourier transform*, provided the integrals on the right hand side . exist. From (5.1) it follows by integration by parts that differentiation

of a function is transformed into multiplication of its Fourier transforms, or an analytical operation is converted into an algebraic operation. More precisely, we have

Proposition 5.1.

$$\widehat{D^\alpha f}(\xi) = i^{|\alpha|} \xi^\alpha \widehat{f}(\xi),$$

where $|\alpha| \leq s$.

The following proposition shows that the Fourier transform of f decreases rapidly for $|\xi| \to \infty$, provided $f \in C_0^s(\mathbb{R}^n)$. In particular, the right hand side of (5.2) exists for $g := \hat{f}$ if $f \in C_0^{n+1}(\mathbb{R}^n)$.

Proposition 5.2. Assume $g \in C_0^s(\mathbb{R}^n)$, then there is a constant $M = M(n, s, g)$ such that

$$|\widehat{g}(\xi)| \leq \frac{M}{(1+|\xi|)^s}.$$

Proof. Let $\xi = (\xi_1, \ldots, \xi_n)$ be fixed and let j be an index such that $|\xi_j| = \max_k |\xi_k|$. Then

$$|\xi| = \left(\sum_{k=1}^n \xi_k^2\right)^{1/2} \leq \sqrt{n}|\xi_j|$$

which implies

$$\begin{aligned}
(1+|\xi|)^s &= \sum_{k=0}^s \binom{s}{k} |\xi|^k \\
&\leq 2^s \sum_{k=0}^s n^{k/2} |\xi_j|^k \\
&\leq 2^s n^{s/2} \sum_{|\alpha| \leq s} |\xi^\alpha|.
\end{aligned}$$

This inequality and Proposition 5.1 imply

$$\begin{aligned}
(1+|\xi|)^s |\widehat{g}(\xi)| &\leq 2^s n^{s/2} \sum_{|\alpha| \leq s} |(i\xi)^\alpha \widehat{g}(\xi)| \\
&\leq 2^s n^{s/2} \sum_{|\alpha| \leq s} \int_{\mathbb{R}^n} |D^\alpha g(x)| \ dx =: M.
\end{aligned}$$

□

The notation *inverse* Fourier transform for (5.2) will be justified by

Theorem 5.1. $\widetilde{\widehat{f}} = f$ *and* $\widehat{\widetilde{f}} = f$.

Proof. See [23], for example. We will prove the first assertion

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\xi \cdot x} \widehat{f}(\xi) \ d\xi = f(x) \tag{5.3}$$

here. The proof of the other relation is left as an exercise. All integrals appearing in the following exist, see Proposition 5.2 and the special choice of g.

(i) Formula

$$\int_{\mathbb{R}^n} g(\xi)\widehat{f}(\xi)\mathrm{e}^{ix\cdot\xi}\ d\xi = \int_{\mathbb{R}^n} \widehat{g}(y)f(x+y)\ dy \tag{5.4}$$

follows by direct calculation:

$$\begin{aligned}
&\int_{\mathbb{R}^n} g(\xi)\left((2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot y}f(y)\ dy\right)\mathrm{e}^{ix\cdot\xi}\ d\xi \\
&\quad = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} g(\xi)\mathrm{e}^{-i\xi\cdot(y-x)}\ d\xi\right)f(y)\ dy \\
&\quad = \int_{\mathbb{R}^n} \widehat{g}(y-x)f(y)\ dy \\
&\quad = \int_{\mathbb{R}^n} \widehat{g}(y)f(x+y)\ dy.
\end{aligned}$$

(ii) Relation

$$(2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{-iy\cdot\xi}g(\varepsilon\xi)\ d\xi = \varepsilon^{-n}\widehat{g}(y/\varepsilon) \tag{5.5}$$

for each $\varepsilon > 0$ follows after substitution $z = \varepsilon\xi$ in the left hand side of (5.1).

(iii) Equation

$$\int_{\mathbb{R}^n} g(\varepsilon\xi)\widehat{f}(\xi)\mathrm{e}^{ix\cdot\xi}\ d\xi = \int_{\mathbb{R}^n} \widehat{g}(y)f(x+\varepsilon y)\ dy \tag{5.6}$$

follows from (5.4) and (5.5). Set $G(\xi) := g(\varepsilon\xi)$, then (5.4) implies

$$\int_{\mathbb{R}^n} G(\xi)\widehat{f}(\xi)\mathrm{e}^{ix\cdot\xi}\ d\xi = \int_{\mathbb{R}^n} \widehat{G}(y)f(x+y)\ dy.$$

Since, see (5.5),

$$\begin{aligned}
\widehat{G}(y) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-iy\cdot\xi}g(\varepsilon\xi)\ d\xi \\
&= \varepsilon^{-n}\widehat{g}(y/\varepsilon),
\end{aligned}$$

we arrive finally at

$$\begin{aligned}\int_{\mathbb{R}^n} g(\varepsilon\xi)\widehat{f}(\xi) &= \int_{\mathbb{R}^n} \varepsilon^{-n}\widehat{g}(y/\varepsilon)f(x+y)\ dy\\ &= \int_{\mathbb{R}^n} \widehat{g}(z)f(x+\varepsilon z)\ dz.\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$g(0)\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{ix\cdot\xi}\ d\xi = f(x)\int_{\mathbb{R}^n} \widehat{g}(y)\ dy. \tag{5.7}$$

Set

$$g(x) := \mathrm{e}^{-|x|^2/2},$$

then

$$\int_{\mathbb{R}^n} \widehat{g}(y)\ dy = (2\pi)^{n/2}. \tag{5.8}$$

Since $g(0) = 1$, the first assertion of Theorem 5.1 folows from (5.7) and (5.8). It remains to show (5.8).

(iv) Proof of (5.8). We will show

$$\begin{aligned}\widehat{g}(y) : &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2/2}\mathrm{e}^{-ix\cdot x}\ dx\\ &= \mathrm{e}^{-|y|^2/2}.\end{aligned}$$

The proof of

$$\int_{\mathbb{R}^n} \mathrm{e}^{-|y|^2/2}\ dy = (2\pi)^{n/2}$$

is left as an exercise. Since

$$-\left(\frac{x}{\sqrt{2}}+i\frac{y}{\sqrt{2}}\right)\cdot\left(\frac{x}{\sqrt{2}}+i\frac{y}{\sqrt{2}}\right) = -\left(\frac{|x|^2}{2}+ix\cdot y-\frac{|y|^2}{2}\right)$$

it follows

$$\begin{aligned}\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2/2}\mathrm{e}^{-ix\cdot y}\ dx &= \int_{\mathbb{R}^n} \mathrm{e}^{-\eta^2}\mathrm{e}^{-|y|^2/2}\ dx\\ &= \mathrm{e}^{-|y|^2/2}\int_{\mathbb{R}^n} \mathrm{e}^{-\eta^2}\ dx\\ &= 2^{n/2}\mathrm{e}^{-|y|^2/2}\int_{\mathbb{R}^n} \mathrm{e}^{-\eta^2}\ d\eta\end{aligned}$$

where
$$\eta := \frac{x}{\sqrt{2}} + i\frac{y}{\sqrt{2}}.$$
Consider first the one-dimensional case. According Cauchy's theorem we have
$$\oint_C \mathrm{e}^{-\eta^2}\ d\eta = 0,$$
where the integration is along the curve C which is the union of four curves as indicated in Figure 5.1.

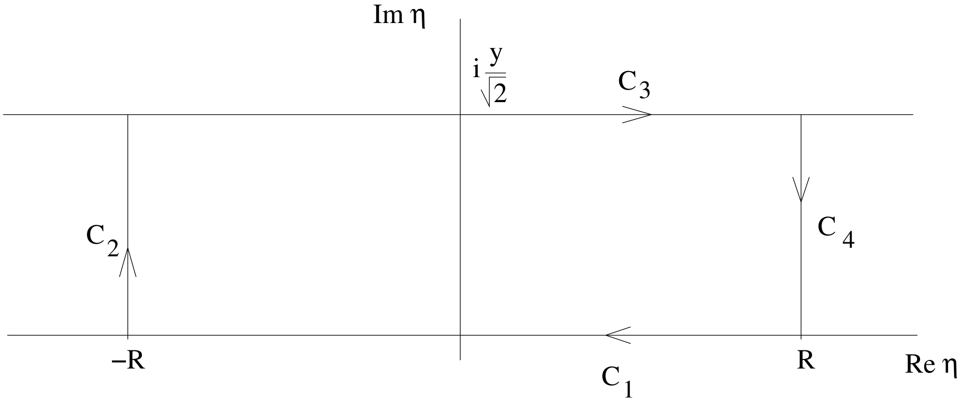

Figure 5.1: Proof of (5.8)

Consequently,
$$\int_{C_3} \mathrm{e}^{-\eta^2}\ d\eta = \frac{1}{\sqrt{2}}\int_{-R}^{R} \mathrm{e}^{-x^2/2}\ dx - \int_{C_2} \mathrm{e}^{-\eta^2}\ d\eta - \int_{C_4} \mathrm{e}^{-\eta^2}\ d\eta.$$
It follows
$$\lim_{R\to\infty}\int_{C_3} \mathrm{e}^{-\eta^2}\ d\eta = \sqrt{\pi}$$
since
$$\lim_{R\to\infty}\int_{C_k} \mathrm{e}^{-\eta^2}\ d\eta = 0, \quad k = 2,\ 4.$$
The case $n > 1$ can be reduced to the one-dimensional case as follows. Set
$$\eta = \frac{x}{\sqrt{2}} + i\frac{y}{\sqrt{2}} = (\eta_1, \ldots, \eta_n),$$

where

$$\eta_l = \frac{x_l}{\sqrt{2}} + i\frac{y_l}{\sqrt{2}}.$$

From $d\eta = d\eta_1 \ldots d\eta_l$ and

$$\mathrm{e}^{-\eta^2} = \mathrm{e}^{-\sum_{l=1}^n \eta_l^2} = \prod_{l=1}^n \mathrm{e}^{-\eta_l^2}$$

it follows

$$\int_{\mathbb{R}^n} e^{-\eta^2}\ d\eta = \prod_{l=1}^n \int_{\Gamma_l} \mathrm{e}^{-\eta_l^2}\ d\eta_l,$$

where for fixed y

$$\Gamma_l = \{z \in \mathbb{C} :\ z = \frac{x_l}{\sqrt{2}} + i\frac{y_l}{\sqrt{2}}, -\infty < x_l < +\infty\}.$$

□

There is a useful class of functions for which the integrals in the definition of $\widehat{f}$ and $\widetilde{f}$ exist.

For $u \in C^\infty(\mathbb{R}^n)$ we set

$$q_{j,k}(u) := \sup_{\mathbb{R}^n} \left\{(1 + |x|^2)^{j/2}|D^\alpha u(x)| :\ |\alpha| \le k\right\}.$$

Definition. The *Schwartz class* of rapidely degreasing functions is

$$\mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) :\ q_{j,k}(u) < \infty \text{ for any } j, k \in \mathbb{N} \cup \{0\}\}$$

This space is a Frechét space.

Proposition 5.3. *Assume $u \in \mathcal{S}(\mathbb{R}^n)$, then $\widehat{u}$ and $\widetilde{u} \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. See [20], Chapter 1.2, for example, or an exercise.

5.1.1 Pseudodifferential operators

The properties of Fourier transform lead to a general theory for, at least linear, partial differential or integral equations. In this subsection we define

$$D_k = \frac{1}{i}\frac{\partial}{\partial x_k}, \quad k = 1, \ldots, n,$$

and for each multiindex α as in Subsection 3.5.1

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}.$$

That is,

$$D^\alpha = \frac{1}{i^{|\alpha|}} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

Let

$$p(x, D) := \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha,$$

where a_α are given sufficiently regular functions, be a linear partial differential of order m.

According to Theorem 5.1 and Proposition 5.3, we have, at least for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} \widehat{u}(\xi)\ d\xi,$$

which implies

$$D^\alpha u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} \xi^\alpha \widehat{u}(\xi)\ d\xi.$$

Consequently

$$p(x, D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} p(x, \xi) \widehat{u}(\xi)\ d\xi, \tag{5.9}$$

where

$$p(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha.$$

The right hand side of (5.9) makes sense also for more general functions $p(x, \xi)$, not only for polynomials.

Definition. The function $p(x, \xi)$ is called *symbol* and

$$(Pu)(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi} p(x, \xi) \widehat{u}(\xi)\ d\xi$$

is said to be *pseudodifferential operator*.

An important class of symbols for which the right hand side in this definition of a pseudodifferential operator is defined is S^m which is the subset of $p(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ such that

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{K,\alpha,\beta}(p)\,(1+|\xi|)^{m-|\alpha|}$$

for each compact $K \subset \Omega$.

Above we have seen that linear differential operators define a class of pseudodifferential operators. Even integral operators can be written (formally) as pseudodifferential operators. Let

$$(Pu)(x) = \int_{textslRn} K(x,y)u(y)\;dy$$

be an integral operator. Then

$$\begin{aligned}(Pu)(x) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} K(x,y)\int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi}\xi^\alpha \widehat{u}(\xi)\;d\xi \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{ix\cdot\xi}\left(\int_{\mathbb{R}^n} \mathrm{e}^{i(y-x)\cdot\xi}K(x,y)\;dy\right)\widehat{u}(\xi).\end{aligned}$$

That is, the symbol associated to the above integral operator is

$$p(x,\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{i(y-x)\cdot\xi}K(x,y)\;dy.$$

5.2 Exercices

1. Show
$$\int_{\mathbb{R}^n} \mathrm{e}^{-|y|^2/2}\ dy = (2\pi)^{n/2}.$$

2. Find a formal solution of Cauchy's initial value problem for the wave equation by using Fourier's transformation.

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